Higher geometry and algebraic K-theory

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Topological K-theory $K^*(-)$ has a geometric interpretation of its (degree 0) cocycles.

- X : finite CW complex
- The set of equivalence classes of complex vector bundles over X:

$$\big[\text{complex vector bundles}/X,\ \oplus\big]$$

  This is a monoid under Whitney sum $\oplus$.
- $\mathrm{Grp}(-)$: the Grothendieck group completion of a monoid.

Then:

$$K^0(X) = \mathrm{Grp}\big[\text{complex vector bundles}/X,\ \oplus\big]$$

$\mathbb{C} \rightsquigarrow R$

Goal: Find a similar description of cocycles for the cohomology theory $K(R)^*(-)$ determined by the algebraic K-theory of a ring spectrum R.

Topological K-theory $K^*(-)$ is the cohomology theory associated to the complex K-theory spectrum K:

- $K = K(\mathbb{C}) =$ Algebraic K-theory(finite rank $\mathbb{C}$-modules)
- $\Omega^\infty K = \mathbb{Z} \times BU \simeq K_0(\mathbb{C}) \times BGL_\infty(\mathbb{C})$

$$\mathbb{C} \rightsquigarrow \text{connective ring spectrum } R$$

The algebraic K-theory spectrum $K(R)$ arises as:

- $K(R) =$ Algebraic K-theory(finite cell R-modules)
- $\Omega^\infty K(R) = K_0(R) \times BGL_\infty(R)^+$

Goal, more precisely

The desired analog of

$$K^0(X) = \mathrm{Grp}\big[\text{complex vector bundles}/X,\ \oplus\big]$$

is:

$$K(R)^0(X) = \mathrm{Grp}\big[\text{bundles of } R\text{-modules}/X,\ \vee\big]$$

$K(R)^*(-)$: cohomology theory associated to the algebraic K-theory spectrum of R.
bundle of R-modules: parametrized family of R-module spectra.

Parametrized Spectra

A parametrized spectrum E over X is a spectrum object $\{E_n\}$ in the category of spaces over and under X:

$$X \xrightarrow{s} E_n \xrightarrow{p} X \qquad p \circ s = \mathrm{id}_X$$

$$\Sigma_X E_n \stackrel{def}{=} E_n \wedge_X S^1_X \xrightarrow{\sigma} E_{n+1} \qquad p \circ \sigma = p, \quad \sigma \circ s = s.$$

In order to employ structured ring and module spectra, we will implicitly use orthogonal spectra and follow the homotopical foundations developed by May-Sigurdsson.

Fiberwise Equivalences

A map $f\colon X \longrightarrow Y$ of base spaces gives rise to a series of base change functors $f_! \dashv f^* \dashv f_*$.

- Pullback along the inclusion of a point $i_x\colon * \longrightarrow X$ determines the fiber of E over $x \in X$:

$$E_x \stackrel{def}{=} i_x^* E$$

- A map $E \longrightarrow E'$ of spectra over X is a fiberwise weak equivalence if the induced map $E_x \longrightarrow E'_x$ on fibers is a weak equivalence of spectra for all $x \in X$.

This is the appropriate notion of equivalence for parametrized spectra.

Untwistings

- If M is a non-parametrized spectrum, we can form the "untwisted" parametrized spectrum

$$M_X = \text{“}M \times X\text{”} = r^* M, \qquad r\colon X \longrightarrow *.$$

  A trivialization of E is a fiberwise equivalence $E \simeq M_X$ for some M.

- For any spectrum E over X, there are associated cohomology groups:

$$E^n(X) = \pi_{-n} r_* F_X(X, E) = \pi_0\{\text{global sections of } E_n \xrightarrow{p} X\}.$$

  More generally, E determines a cohomology theory on the category of spaces over X. When $E \simeq M_X$ is trivial,

$$E^*(X) = M^*(X).$$

R-bundles

Let R be a connective ring spectrum.

- An R-bundle E over X is a parametrized spectrum with an associative and unital action of R over X

$$R \wedge E \longrightarrow E.$$

Each fiber E_x is a (non-parametrized) R-module spectrum.

Examples

- Given a twisted coefficient system $\pi_1 X \circlearrowright V$, there is a parametrized Eilenberg MacLane spectrum HV over X. The associated cohomology is ordinary cohomology with coefficients in V:

$$HV^*(X) = H^*(X; V).$$

- Let $\tau \in H^3(X; \mathbb{Z})$. The τ-twisted K-theory $K^*_\tau(X)$ of X arises as the cohomology associated to a K-bundle $E(\tau)$ over X with fiber K:

$$E(\tau)^*(X) = K^*_\tau(X).$$

- Given a link diagram D, there is an $H\mathbb{Z}$-bundle E_D over the space X_D of "crossing data" of D. The associated cohomology is the Khovanov homology of D:

$$E_D^*(X_D) = KH^*(D) \qquad \text{[Everitt-Turner]}$$

Examples

- Applying Σ^∞_X to the free loopspace fibration

$$\Omega X \longrightarrow LX \xrightarrow{ev} X,$$

we have a parametrized spectrum $\Sigma^\infty_X LX$ over X with fiber $\Sigma^\infty_+ \Omega X$.

- The Cohen-Jones ring spectrum LM^{-TM} (whose homology realizes the Chas-Sullivan string product) arises from a parametrized spectrum ELM^{-TM} over M:

$$LM^{-TM} = r_! ELM^{-TM}, \qquad r\colon M \longrightarrow *.$$

T. Kragh: Although ELM^{-TM} is non-trivially twisted, its homology is untwisted. From this he deduces a homotopy theoretic form of the nearby Lagrangian conjecture in symplectic topology.

The Main Theorem

- An R-bundle E has finite rank if every fiber admits an equivalence $E_x \simeq R^{\vee n}$ for some $n \geq 0$ (possibly varying over the components of X).

Main Theorem

Let R be a connective ring spectrum and let X be a finite CW complex. There is a natural isomorphism

$$K(R)^0(X) \cong Grp[\text{virtual finite rank } R\text{-bundles}/X, \vee].$$

"virtual" means that we pass to homologically equivalent covers of X when considering bundle classes. This is forced by the non-trivial effect of Quillen's plus construction.

The Main Theorem: $R = ku$ and 2-vector bundles

Let $R = ku$ be the connective complex K-theory spectrum.

Main Theorem ($R = ku$)

$$K(ku)^0(X) \cong Grp[\textit{virtual finite rank ku-bundles}/X, \ \vee].$$

- Instead of the analogy $\mathbb{C} \rightsquigarrow ku$, we could try:

  $\mathbb{C} \rightsquigarrow (\mathrm{Vect}_{\mathbb{C}}, \oplus, \otimes) =$ the "ring category" of finite rank $\mathbb{C}$ v.s.

- Baas-Dundas-Richter-Rognes define the algebraic K-theory $K(\mathrm{Vect}_{\mathbb{C}})$ and give a similar description of $K(\mathrm{Vect}_{\mathbb{C}})^0(X)$ in terms of bundles of $\mathrm{Vect}_{\mathbb{C}}$-module categories over X.
- The equivalence $K(ku) \simeq K(\mathrm{Vect}_{\mathbb{C}})$ [BDRR, Osorno] means that ku-bundles and 2-vector bundles provide two geometric descriptions of the same cohomology theory.

The Classification Theorem

- $\mathrm{End}_R M$: the A_∞-space of R-module maps $M \longrightarrow M$
- $\mathrm{Aut}_R M = \mathrm{GL}_1 F_R(M, M)$: the grouplike A_∞-space of R-module equivalences $M \longrightarrow M$
- For example,

$$\mathrm{GL}_n R \stackrel{def}{=} \mathrm{Aut}_R(R^{\vee n}).$$

The main theorem follows from:

The Classification Theorem

The space $B\mathrm{Aut}_R M$ classifies R-bundles with fiber M.
More precisely, there is a natural isomorphism of equivalence classes:

$$[X, B\mathrm{Aut}_R M] \cong [R\text{-bundles over } X \text{ with fiber } M].$$

The Classification Theorem

The Classification Theorem

The space $BAut_R M$ classifies R-bundles with fiber M. More precisely, there is a natural isomorphism of equivalence classes:

$$[X, BAut_R M] \cong [R\text{-bundles over } X \text{ with fiber } M].$$

Ando-Blumberg-Gepner take this as their starting point: using the language of quasicategories, they *define* the space $Map(X, B\mathrm{Aut}_R M)$ to be the $(\infty, 1)$-category of R-bundles with fiber M. Their comparison with May-Sigurdsson should specialize to give a version of this result.

Thom Spectra arise as line R-bundles

Ando-Blumberg-Gepner-Hopkins-Rezk: Given a map

$$f\colon X \longrightarrow BGL_1 R,$$

we can form the parametrized line R-bundle Lf over X corresponding to f by the classification theorem. The R-module Thom spectrum Mf associated to f is:

$$Mf = r_! Lf \qquad r\colon X \longrightarrow *$$

The Lf cohomology of X defines the f-twisted R-theory of X:

$$Lf^*(X) = R^*_f(X).$$

Proof of the Classification Theorem

Underlying technology (originates in work of Bökstedt/EKMM):

Theorem (Blumberg, L., Schlichtkrull-Sagave)

There is a symmetric monoidal model category $(\mathcal{A}, \boxtimes, *)$ *with* $\mathrm{Ho}\mathcal{A} \simeq \mathrm{Ho}\,\mathit{Top}$ *such that:*

$$\{\boxtimes\text{-monoids in } \mathcal{A}\} \simeq \{A_\infty\text{-spaces}\}$$
$$\{\text{commutative } \boxtimes\text{-monoids in } \mathcal{A}\} \simeq \{E_\infty\text{-spaces}\}$$

Let G be a grouplike $\boxtimes$-monoid in $\mathcal{A}$. Using the two-sided bar construction built out of $\boxtimes$, we can define a "universal principal G-bundle"

$$EG = B^{\boxtimes}(G, G, *) \longrightarrow B^{\boxtimes}(*, G, *) = BG.$$

Pullback of EG along a map $f\colon X \longrightarrow BG$ induces:

$$[X, BG] \cong [\text{principal } G\text{-bundles}/X].$$

Proof of the Classification Theorem

When $G = \mathrm{Aut}_R M$, this is:

$$[X, B\mathrm{Aut}_R M] \cong [\text{principal } \mathrm{Aut}_R M\text{-bundles}/X].$$

Forming the associated R-bundle with fiber M

$$Y \longmapsto M \wedge_{\Sigma^\infty_+ \mathrm{Aut}_R M} \Sigma^\infty_X Y$$

induces a natural isomorphism:

$$[\text{principal } \mathrm{Aut}_R M\text{-bundles}/X] \cong [R\text{-bundles over } X \text{ with fiber } M].$$