NOTES FROM A TALK ON PARAMETRIZED THOM SPECTRA AND ORIENTATION THEORY

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An n-dimensional vector bundle $\xi\colon V \longrightarrow X$ gives rise to a spherical fibration $S^\xi\colon S^V \longrightarrow X$, and thus to a local coefficient system

$$\widetilde{H}^*(S^V_\bullet)\colon \Pi_1 X \longrightarrow \mathrm{grAb}$$
$$p \in X \longmapsto \widetilde{H}^*(S^V_p)$$

An orientation of ξ is an isomorphism of local coefficient systems $\widetilde{H}^*(S^V_\bullet) \cong \underline{\mathbf{Z}}[n]$, where $\underline{\mathbf{Z}}[n]$ is the constant system given by the integers in degree n and 0 elsewhere. The Serre spectral sequence associated with $\widetilde{H}^*(S^V_\bullet)$ takes the form

$$H^p(X, \widetilde{H}^q(S^V_\bullet)) \Longrightarrow H^{p+q}(S^V, X) \cong \widetilde{H}^{p+q}(X^\xi)$$

where, $X^\xi = S^V/X$ is the Thom space of ξ.

An orientation of ξ "untwists" this spectral sequence so that it converges to $H^{p+q-n}(X)$. The resulting isomorphism of E_∞ terms is the Thom isomorphism in integral cohomology.

Let $\gamma(n)$ be the tautological n-plane bundle over $BO(n)$. Then the Thom space $BO(n)^{\gamma(n)}$ is the n-th space of the Thom spectrum MO representing unoriented corbodism theory. More generally, given a compatible system of maps $f_n\colon X_n \longrightarrow BO(n)$, we may define a spectrum Mf whose n-th space is $X_n^{f_n^*\gamma(n)}$. Using this method, we can construct $MSO, MSp, MSpin, MString, MU$, etc. If $\xi - \eta$ is a virtual vector, bundle, we choose ξ' such that $\eta \oplus \xi'$ is the trivial bundle ϵ_N of rank N, and then define:

$$X^{\xi-\eta} = \Sigma^N X^{\xi \oplus \xi'}.$$

Alternatively, we could pass to the colimit of classifying spaces. The space $BO = \mathrm{colim}_n BO(n)$ classifies rank zero virtual vector bundles, and we define the Thom spectrum associated to a map $f\colon X \longrightarrow BO$ to be the Thom spectrum Mf associated with the system $f_n\colon f^{-1}BO(n) \longrightarrow BO(n)$ of maps into the finite skeleta.

In fact, the construction of Thom spectra only depends on the associated spherical fibration of a vector bundle. Let $h\mathrm{Aut}(S^n)$ be the monoid of based homotopy equivalences $S^n \longrightarrow S^n$. Then the classifying space $Bh\mathrm{Aut}(S^n)$ classifies fibrations with fiber S^n. The J-homomorphism $J\colon BO(n) \longrightarrow Bh\mathrm{Aut}(S^n)$ is induced by one-point compactification. We may think of the space $\Omega^\infty S = \mathrm{colim}_n \Omega^n S^n$ of stable self-maps of spheres as the space $\mathrm{Hom}_S(S, S)$ of S-module endomorphisms of the sphere spectrum. The subspace $\mathrm{colim}_n h\mathrm{Aut}(S^n) \subset \Omega^\infty S$ corresponds to the space $\mathrm{GL}_1 S$ of S-module automorphisms of S.

Suppose we are given a (not necessarily commutative) S-algebra R with unit map $\eta\colon S \longrightarrow R$. There is an A_∞ space of units $\mathrm{GL}_1 R \subset \Omega^\infty R$ corresponding to the subspace of $\mathrm{Hom}_R(R, R)$ consisting of R-module automorphisms. Composing the J-homomorphism with the map of units induced by η, we have the diagram

$$BO \xrightarrow{J} \mathrm{colim}_n Bh\mathrm{Aut}(S^n) = B\mathrm{GL}_1 S \longrightarrow B\mathrm{GL}_1 R$$

We will now extend the construction of Thom spectra to accept maps $f\colon X \longrightarrow B\mathrm{GL}_1 R$ as input.

To this end, we define a universal principal $\mathrm{GL}_1 R$-bundle $E\mathrm{GL}_1 R \longrightarrow B\mathrm{GL}_1 R$ in terms of a two-sided bar construction

$$B(*, \mathrm{GL}_1 R, \mathrm{GL}_1 R) \longrightarrow B(*, \mathrm{GL}_1 R, *).$$

In order to do this truthfully, one needs to make a choice of model for A_∞ spaces as monoids in some category with a symmetric monoidal structure $\boxtimes$. One then forms the bar construction in the usual way but with respect to $\boxtimes$ instead of the cartesian product. See [5, 7, 9] for a few different approaches to the required technology.

Writing B for $B\mathrm{GL}_1 R$, we may form the parametrized suspension spectrum $\Sigma^\infty_B E\mathrm{GL}_1 R$, whose fibers are of the form $\Sigma^\infty_+ E\mathrm{GL}_1 R$. In particular, this is a right $\Sigma^\infty_+ \mathrm{GL}_1 R$-module. The spectrum R is a left $\Sigma^\infty_+ \mathrm{GL}_1 R$-module. In anology with the vector bundle associated to the universal principal $O(n)$-bundle, we define the universal rank 1 R-module bundle (a.k.a. line R-bundle) to be the parametrized R-module

$$\Sigma^\infty_B E\mathrm{GL}_1 R \wedge_{\Sigma^\infty_+ \mathrm{GL}_1 R} R.$$

Definition. Let $f\colon X \longrightarrow B\mathrm{GL}_1 R$ be a map of spaces. The parametrized Thom spectrum associated to f is the base change of the universal rank 1 R-module bundle along the map f:

$$\begin{aligned} M_X f &= f^*(\Sigma^\infty_B E\mathrm{GL}_1 R \wedge_{\Sigma^\infty_+ \mathrm{GL}_1 R} R) \\ &\cong \Sigma^\infty_X f^* E\mathrm{GL}_1 R \wedge_{\Sigma^\infty_+ \mathrm{GL}_1 R} R. \end{aligned}$$

$M_X f$ is a rank 1 R-bundle over X, i.e. a parametrized R-module spectrum over X whose fibers are equivalent to R. The total Thom spectrum associated to f is the R-module$Mf = r_! M_X f$. Here, $r_!$ is the left adjoint to the functor r^* from spectra to parametrized spectra that gives the "untwisted" parametrized spectrum. It is the base change functor associated to the map $r\colon X \longrightarrow *$.

By the definition of $M_X f$, we have the description:

$$Mf = r_! f^*(\Sigma^\infty_B E\mathrm{GL}_1 R \wedge_{\Sigma^\infty_+ \mathrm{GL}_1 R} R) \cong \Sigma^\infty_+ f^* E\mathrm{GL}_1 R \wedge_{\Sigma^\infty_+ \mathrm{GL}_1 R} R.$$

This is the definition for Mf described in [2]. When $S = R$, this agrees with the old definition of the Thom spectrum associated to f. If f factors through a map $g\colon X \longrightarrow B\mathrm{GL}_1 S$, then we have an isomorphism of parametrized Thom spectra $M_X f \cong M_X g \wedge R$.

The following theorem justifies the use of the word "universal" above:

Theorem. [8] *The association*

$$(f\colon X \longrightarrow B\mathrm{GL}_1 R) \mapsto M_X f$$

induces a bijection between the set of homotopy classes of maps $[X, B\mathrm{GL}_1 R]$ and the set of fiberwise weak equivalence classes of parametrized rank one R-module spectra over X.

Suppose that E is a parametrized spectrum with fiber M. We say that E is trivializable if there is a weak homotopy equivalence of parametrized spectra $E \simeq r^* M$. It follows from the theorem that the Thom spectrum $M_X f$ is trivializable if and only if the the map $f\colon X \longrightarrow B\mathrm{GL}_1 R$ null-homotopic.

From the quasicategory point of view, there is a model for $B\mathrm{GL}_1 R$ that suggests we take this theorem as a definition. In [1], a parametrized rank 1 R-module spectrum over X is defined to be a map $f\colon X \longrightarrow B\mathrm{GL}_1 R$.

Theorem (Mahowald-Ray)**.** *Let $\xi\colon Y \longrightarrow X$ be a spherical fibration over X, and let $f(Y) \longrightarrow B\mathrm{GL}_1 S \longrightarrow B\mathrm{GL}_1 R$ be the map induced by the classifying map for Y. The spherical fibration Y is R-orientable if and only if the parametrized Thom spectrum $M_X f(Y)$ is trivializable.*

Proof. A Thom class $\mu \in R^n(X^\xi)$ may be represented by a map $\mu\colon r_! Y = X^\xi \longrightarrow \Sigma^n R$ with adjoint $\widetilde{\mu}\colon Y \longrightarrow r^* \Sigma^n R = R \wedge_X S^n_X$. Composing with the multiplication of R gives a map

$$\psi\colon M_X f = R \wedge_X Y \xrightarrow{\mathrm{id} \wedge \widetilde{\mu}} R \wedge R \wedge_X S^n_X \longrightarrow R \wedge_X S^n_X.$$

For each point $x \in X$, the class $\mu_x \in R^n(Y_x) \cong R^n(S^n)$ is a unit if and only if the restriction ψ_x of ψ to the map of fibers over x is a weak equivalence of R-modules. This proves the theorem. □

When ξ is R-oriented, we may deduce the Thom isomorphisms

$$R_*(X_+) \cong R_{*+n}(X^\xi) \qquad R^*(X_+) \cong R^{*+n}(X^\xi)$$

from the equivalences of spectra

$$R \wedge X^\xi \simeq R \wedge \Sigma^n X_+ \qquad F(X^\xi, R) \simeq F(\Sigma^n X_+, R)$$

obtained by applying $r_!$ to the equivalence of parametrized spectra given in the theorem.

Example. Let $R = H\mathbf{Z}$. Then $\mathrm{GL}_1 H\mathbf{Z} = \mathbf{Z}/2$, and the composite

$$w_2 \colon BO \xrightarrow{J} B\mathrm{GL}_1 S \longrightarrow B\mathrm{GL}_1 H\mathbf{Z} = K(\mathbf{Z}/2, 1)$$

represents the first Stiefel-Whitney class w_2. Let ξ be a vector bundle, and let $f \colon X \longrightarrow BO$ be the map induced by the map representing ξ. The vector bundle ξ is $H\mathbf{Z}$-orientable if and only if the Thom spectrum $M_X f = X^\xi \wedge H\mathbf{Z}$ is trivializable. The latter condition is equivalent to the vanishing of the first Stiefel Whitney class $w_1(\xi) = [w_1 \circ f] \in H^1(X; \mathbf{Z}/2)$.

Example. Let $R = K$ be complex K-theory. Then $\Omega^\infty K = \mathbf{Z} \times BU$, and the space of units of K decomposes as a product

$$\mathrm{GL}_1 K = \mathbf{Z}/2 \times BU(1) \times BSU_\otimes = \mathbf{Z}/2 \times K(\mathbf{Z}, 2) \times BSU_\otimes.$$

If we pass to the connected cover SO of O, the map

$$SO \xrightarrow{J} \mathrm{GL}_1 S \longrightarrow \mathrm{GL}_1 K$$

factors through the connected cover $\mathrm{SL}_1 K = K(\mathbf{Z}, 2) \times BSU_\otimes$ of $\mathrm{GL}_1 K$. At the level of classifying spaces, the map

$$w_2 \colon BSO \xrightarrow{J} B\mathrm{GL}_1 S \longrightarrow B\mathrm{GL}_1 K = K(\mathbf{Z}/2, 1) \times K(\mathbf{Z}, 3) \times BBSU_\otimes \xrightarrow{\pi} K(\mathbf{Z}/2, 1)$$

representing the first Stiefel-Whitney class is nullhomotopic. Let $Spin(n)$ be the universal cover of $SO(n)$, realized as a Lie group, and let

$$Spin^c(n) = Spin(n) \times_{\mathbf{Z}/2} U(1)$$

be the associated principal $U(1)$-bundle over $SO(n)$. Then $Spin^c(n)$ is also a compact Lie group, and the colimit of the classifying spaces $BSpin^c = \mathrm{colim}_n\, BSpin^c(n)$ is the fiber of the composite

$$BSO \xrightarrow{w_2} K(\mathbf{Z}/2, 2) \xrightarrow{\beta} K(\mathbf{Z}, 3) = BU(1)$$

of the second Stiefel-Whitney class and the Bockstein β. Therefore we have the following commutative diagram.

$$\begin{array}{ccccc}
BSpin^c & & \longrightarrow & & E\mathrm{GL}_1 K \simeq * \\
\downarrow & & & & \downarrow \\
BSO & \xrightarrow{J} & B\mathrm{GL}_1 S & \longrightarrow & B\mathrm{GL}_1 K \\
& \searrow^{\beta w_2} & & & \downarrow \pi \\
& & & & K(\mathbf{Z}, 3)
\end{array}$$

Since the map from $BSpin^c$ to $B\mathrm{GL}_1 K$ is nullhomotopic, it follows that a real vector bundle ξ is K-orientable if and only if it has a reduction of its structural group to $Spin^c$, i.e. if $w_1(\xi) = 0$ and $\beta w_2(\xi) = 0$. This is the result of Atiyah-Bott-Shapiro [3], who constructed Thom isomorphisms in K-theory for $Spin^c$-bundles.

Let $f \colon BSpin^c \longrightarrow B\mathrm{GL}_1 K$ be the composite in the diagram. Then f is nullhomotopic, so the parametrized Thom spectrum $M_{BSpin^c} f$ is trivializable:

$$M_{BSpin^c} f \simeq S_{BSpin^c} \wedge K.$$

Applying $r_!$ to the trivialization yields an equivalence of ring spectra (the Thom isomorphism):

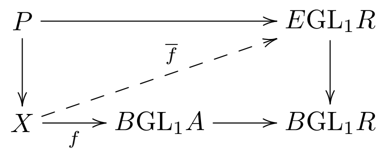
$$MSpin^c \wedge K \simeq BSpin^c_+ \wedge K.$$

In modern language, the Atiyah-Bott-Shapiro orientation is the map of ring spectra $MSpin^c \longrightarrow K$ given by the composite

$$MSpin^c \longrightarrow MSpin^c \wedge K \simeq BSpin^c_+ \wedge K \longrightarrow K$$

of the unit for K theory with the projection of $BSpin^c$ to a point. See [6] for a direct construction of the map of ring spectra. We can think about the orientation map geometrically. The homotopy of $MSpin^c$ is the ring of bordism classes of manifolds equipped with $Spin^c$-structures on their tangent bundles, and the homotopy of the complex K-theory spectrum is given by equivalence classes of complex Hilbert spaces with an action of the Clifford algebra $\mathrm{Cliff}(\mathbf{C}^n)$ and an odd skew-adjoint $\mathrm{Cliff}(\mathbf{C}^n)$-linear Fredholm operator. The Atiyah-Bott-Sapiro orientation sends such a manifold M to the Hilbert space of L^2 sections of the spinor bundle, equipped with a Cliff($\mathbf{C}^n$)-action, along with the Fredholm operator given by the Dirac operator constructed from the connection associated to a choice of metric on M.

In general, if A is a ring spectrum and R is an A-algebra, we define an R-orientation of the Thom spectrum Mf associated to $f\colon X \longrightarrow B\mathrm{GL}_1 A$ to be a choice of lift $\overline{f}$ in the following diagram:



In other words, $\overline{f}$ is a choice of trivialization of the rank one R-bundle associated to f. A choice of orientation is equivalent to a choice of a map of A-algebra spectra $Mf \longrightarrow R$ that can be thought of as the "projection to the fiber." This is the approach to orientations developed by Ando, Blumberg, Gepner, Hopkins, and Rezk [2, 4] in order to calculate the space of tmf-orientations of $MString$.

References

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