NOTES ON PARAMETRIZED SPECTRA

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These notes are meant to be an easy but honest account of the foundations of parametrized stable homotopy theory. I use the book [1] by May-Sigurdsson as the primary reference.

1. Parametrized spaces and base-change functors

Let B be a topological space. We write $\mathcal{T}\mathrm{op}/B$ for the category of spaces $(X, p) = (p\colon X \longrightarrow B)$ over B, and we write $\mathcal{T}\mathrm{op}_B$ for the category of ex-spaces, by which we mean spaces $(X, p, s) = (B \xrightarrow{s} X \xrightarrow{p} B)$ over B equipped with a section: $p \circ s = \mathrm{id}_B$. We write $(X, p)_+ = (X \sqcup B, p \sqcup \mathrm{id}_B, i_B)$ for the ex-space obtained from a parametrized space (X, p) by adjoinging a disjoint section. Ex-spaces are the parametrized analog of based-spaces and the functor $(X, p) \longmapsto (X, p)_+$ is formally analogous to the disjoint basepoint functor $X \longmapsto X_+$.

Let $f\colon A \longrightarrow B$ be a map of spaces. Associated to f are base-change functors

$$\begin{aligned} f_!\colon \mathcal{T}\mathrm{op}_A &\longrightarrow \mathcal{T}\mathrm{op}_B \\ f^*\colon \mathcal{T}\mathrm{op}_B &\longrightarrow \mathcal{T}\mathrm{op}_A \qquad \text{that satisfy } f_! \dashv f^* \dashv f_* \\ f_*\colon \mathcal{T}\mathrm{op}_A &\longrightarrow \mathcal{T}\mathrm{op}_B \end{aligned}$$

i.e. $f_!$ is left adjoint to f^* and f^* is left adjoint to f_*. The pushforward $f_!(X, p, s)$ of an ex-space X along f is defined by the pushout

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle s}\downarrow & & \downarrow \\ X & \longrightarrow & f_!X. \end{array}$$

In other words, the space $f_!X$ is obtained by glueing a copy of B to X along the section $s(A) \subset X$. This defines the section $B \longrightarrow f_!X$. The projection $f_!X \longrightarrow B$ is induced by the projection map of X. Notice that when the section is its own connected component, the pushforward has a very simple description:

$$f_!(X, p)_+ = (X, f \circ p)_+.$$

The pullback $f^*(X, p, s)$ of an ex-space X along f is defined by the pullback square

$$\begin{array}{ccc} f^*X & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle p} \\ A & \xrightarrow{f} & B \end{array}$$

with section $A \longrightarrow f^*X$ induced by the section of X. The right adjoint f_* of f^* is a little harder to construct. We must first construct the internal hom $\mathrm{Map}_B(-, -)$, or parametrized mapping space, of a pair of parametrized spaces. This functor satisfies the adjunction

$$\mathcal{T}\mathrm{op}/B(X \times_B Y, Z) \cong \mathcal{T}\mathrm{op}/B(X, \mathrm{Map}_B(Y, Z))$$

and makes $\mathcal{T}\mathrm{op}/B$ into a cartesian closed category. To construct $\mathrm{Map}_B(-,-)$, we first consider the partial map classifier $\widetilde{Y}$ of a space Y. As a set, $\widetilde{Y} = Y \sqcup \{\omega\}$. We endow $\widetilde{Y}$ with the topology whose closed sets are $\widetilde{Y}$ and the closed sets of Y. This definition is arranged so that a map $f\colon K \longrightarrow \widetilde{Y}$ defined on a closed subset $K \subset X$ is equivalent to the specification of a map $\widetilde{f}\colon X \longrightarrow \widetilde{Y}$ (points outside of K are sent to ω). We then define the mapping ex-space $\mathrm{Map}_B(X, Y)$ from (X, p) to (Y, q) to be the pullback

$$\begin{array}{ccc} \mathrm{Map}_B(X,Y) & \longrightarrow & \mathrm{Map}(X,\widetilde{Y}) \\ \downarrow & & \downarrow {\scriptstyle \mathrm{Map}(X,\widetilde{q})} \\ B & \xrightarrow{\lambda} & \mathrm{Map}(X,\widetilde{B}) \end{array}$$

where

$$\lambda(b)\colon x \longmapsto \begin{cases} b & \text{if } p(x) = b, \\ \omega & \text{otherwise.} \end{cases}$$

In other words, λ is adjoint to the map $X \times B \longrightarrow \widetilde{B}$ classifying $\mathrm{graph}(p) \xrightarrow{\pi_2} B$.

With the parametrized mapping space $\mathrm{Map}_B(-,-)$ defined, we may now define $f_*(X, p, s)$ to be the pullback

$$\begin{array}{ccc} f_*X & \longrightarrow & \mathrm{Map}_B(A,X) \\ \downarrow & & \downarrow {\scriptstyle \mathrm{Map}_B(A,p)} \\ B & \longrightarrow & \mathrm{Map}_B(A,A) \end{array}$$

where the bottom map is adjoint to $\mathrm{id}_A\colon B \times_B A \cong A \longrightarrow A$. Unfortunately, the construction of $\widetilde{Y}$, and hence of $\mathrm{Map}_B(X, Y)$, takes us outside of the category of compactly generated spaces (= weak Hausdorff + k-spaces). If we start with k-spaces, then $\mathrm{Map}_B(X, Y)$ is again a k-space. Therefore, to have a theory of parametrized spaces for which there is always a right adjoint f_* to the pullback functor f^*, one solution is to remove the weak Hausdorff condition and work only with k-spaces. This is the approach followed in May-Sigurdsson's book, and so by "$\mathcal{T}$op" we mean the category of k-spaces.

We now consider important examples of constructions given by base-change functors. We will always write $r\colon B \longrightarrow \star$ for the projection from B to a point. The base-change functors associated to r are given by collapsing the section

$$r_!(X, p, s) = X/s(B), \qquad \text{so in particular} \quad r_!(X, p)_+ = X_+,$$

by the product space over B

$$r^*X = (X \times B, \pi_2, (*_X, \mathrm{id}_B)),$$

and by the global sections functor

$$r_*(X, p, s) = \{f\colon B \longrightarrow X \mid p \circ f = \mathrm{id}_B\} = \Gamma_B X.$$

Let $i_b\colon * \longrightarrow B$ denote the inclusion of a given point $b \in B$. Then pullback along i_b specifies the fiber of a parametrized space over b:

$$i_b^* X = X_b.$$

Since i_b^* is both a left and a right adjoint, this means that the fiber functor commutes with all colimits and all limits.

Suppose that (X, p, s) and (Y, q, t). are ex-spaces over B. The fiberwise smash-product $X \wedge_B Y$ is defined by the pushout:

$$\begin{array}{ccc} X \vee_B Y & \xrightarrow{(\mathrm{id}_X, t)\vee(s, \mathrm{id}_Y)} & X \times_B Y \\ {\scriptstyle p\vee q}\downarrow & & \downarrow \\ B & \longrightarrow & X \wedge_B Y \end{array}$$

The definition is arranged so that $(X \wedge_B Y)_b = X_b \wedge Y_b$. The category $\mathcal{T}\mathrm{op}_B$ of ex-spaces is symmetric monoidal under $\wedge_B$ with unit object the parametrized 0-sphere $r^* S^0 = S^0_B = (B, \mathrm{id}_B)_+$.

2. Model category structures

We say that a map $X \longrightarrow Y$ in $\mathcal{T}\mathrm{op}/B$ or $\mathcal{T}\mathrm{op}_B$ is a weak equivalence if it induces a weak homotopy equivalence $\pi_* X \longrightarrow \pi_* Y$ of the underlying spaces. Notice that by the five-lemma, this is equivalent to the condition that for every $b \in B$ the induced map of fibers $X^{\mathrm{fib}}_b \longrightarrow Y^{\mathrm{fib}}_b$ of a fibrant approximation $X \xrightarrow{\simeq} X^{\mathrm{fib}} \twoheadrightarrow B$ over B is a weak homotopy equivalence.

Recall that the Quillen model category structure on $\mathcal{T}\mathrm{op}$ has weak equivalences the weak homotopy equivalences, fibrations the Serre fibrations, and cofibrations the relative I-cell complexes, with generating cofibrations and acyclic cofibrations

$$I = \{i\colon S^{q-1} \longrightarrow D^q\} \quad \text{and} \quad J = \{j\colon D^q \longrightarrow D^q \times I\}.$$

In other words, a cofibration is an inclusion determined by a series of attachments of cells built by pushout along the maps i. Since $\mathcal{T}\mathrm{op}/B$ and $\mathcal{T}\mathrm{op}_B$ are comma categories, they carry induced model category structures with weak equivalences, fibrations, and cofibrations determined by the forgetful functor to $\mathcal{T}\mathrm{op}$.

May-Sigurdsson do *not* use the comma model category structures. Their reasons are quite subtle, and I will discuss them below. The alternative model structure (which they call the qf-model structure) on $\mathcal{T}\mathrm{op}/B$ has weak equivalences determined by the forgetful functor to $\mathcal{T}\mathrm{op}$, and has generating cofibrations given by the set I_B of maps over B of the form

$$\begin{array}{ccc} S^{q-1} & \xrightarrow{\;i\;} & D^q \\ & \searrow \quad \swarrow & \\ & B & \end{array}$$

where the inclusion i is required to look sufficiently like a projection in a collar neighborhood of ∂D^q so that it is a fiberwise cofibration, meaning a map that satisfies the homotopy extension property in the category $\mathcal{T}\mathrm{op}/B$. If this condition were ommited, and we took all maps $i\colon S^{q-1} \longrightarrow D^q$ with arbitrary projection to B, the resulting set would generate the comma model structure on $\mathcal{T}\mathrm{op}/B$. The set J_B of generating acyclic cofibrations is similarly defined in terms of a condition on the map from the disk to the base space. By putting restrictions on the cofibrations in the mode structure, there are necessarily more fibrations. The fibrations in the qf-model structure are more general than Serre fibrations, but are still quasifibrations, so they admit a long exact sequence in homotopy groups.

There is an induced qf-model structure on the category $\mathcal{T}\mathrm{op}_B$ of ex-spaces detemined by the forgetful functor $\mathcal{T}\mathrm{op}_B \longrightarrow \mathcal{T}\mathrm{op}/B$. The generating cofibrations I_B^+ and acyclic cofibrations J_B^+ are built from those in I_B and J_B by adjoining a disjoint section.

3. Parametrized spectra

A spectrum X over B is an orthogonal spectrum object in the pointed category $\mathcal{T}\mathrm{op}_B$. Thus X consists of an $O(n)$-equivariant ex-space $(X(n), p(n), s(n))$ for each $n \geq 0$, along with $(O(m) \times O(n))$-equivariant spectrum structure maps

$$\sigma_{m,n}\colon S_B^m \wedge_B X(n) \longrightarrow X(m+n) \quad \text{in } \mathcal{T}\mathrm{op}_B$$

where $S_B^m = (B \times S^m, \pi_1, (\mathrm{id}_B, *))$ is the product ex-space over B with fiber the sphere S^m, and $\wedge_B$ denotes the fiberwise smash product. A map of spectra over B is a map $X(n) \longrightarrow Y(n)$ of ex-spaces for each n respecting the structure maps and orthogonal group actions. We write $\mathcal{S}\mathrm{p}_B$ for the category of spectra over B. The definition of a parametrized spectrum is arranged so that for each $b \in B$, the fibers $X(n)_b$ assemble into a non-parametrized spectrum X_b, called the fiber spectrum. We write $\Sigma_B^\infty Y = \Sigma_B^\infty(Y, p, s)$ for the fiberwise suspension spectrum of an ex-space (Y, p, s), defined by $(\Sigma_B^\infty Y)(n) = S_B^n \wedge_B Y$, and write $\Sigma_B^\infty(Y, p)_+$ for the fiberwise suspension spectrum of the ex-space $(Y \sqcup B, p \sqcup \mathrm{id}_B, \mathrm{id}_B)$ obtained from (Y, p) by adjoining a disjoint section.

We use orthogonal spectra so that there is a symmetric monoidal fiberwise smash product $- \wedge_B -$ of spectra over B. Explicitly, this is the coequalizer of the actions of the sphere spectrum $S_B = \Sigma_+^\infty(B, \mathrm{id})_+$ on X and Y

$$X \wedge_B^O S \wedge_B^O Y \rightrightarrows X \wedge_B^O Y \longrightarrow X \wedge_B Y,$$

where $\wedge_B^O$ denotes the fiberwise smash product of orthogonal sequences:

$$(X \wedge_B^O Y)(k) = \bigvee_{m+n=k} O(k)_+ \wedge_{O(m)\times O(n)} (X(m) \wedge_B Y(n)).$$

This is the same definition as for orthogonal spectra, but performed fiberwise over B. The unit object for the fiberwise smash product $\wedge_B$ of spectra is the sphere spectrum $S_B = r^*S = \Sigma_+^\infty(B, \mathrm{id})_+$ over B.

May-Sigurdsson construct a stable model structure on the category of parametrized spectra over B. The weak equivalences are the stable equivalences. These are maps $X \longrightarrow Y$ of spectra over B which induce a weak homotopy equivalence of fiber spectra after taking a level-wise approximation by fibrations:

$$X \xrightarrow{\simeq} Y \quad \Longleftrightarrow \quad \pi_* X_b^{\mathrm{level-fib}} \xrightarrow{\cong} \pi_* Y_b^{\mathrm{level-fib}} \; \forall b \in B.$$

The set I_B^Σ of generating cofibrations is obtained by applying $\Sigma_B^{\infty \pm n}$ to the generating cofibrations in I_B^+ for all n, and similarly for the set of generating acyclic cofibrations J_B^Σ. The fibrations are then determined by the weak equivalences and cofibrations, and are fiberwise Ω-spectra whose projections to B are all qf-fibrations. It is in the verification of the model category axioms for $\mathcal{S}\mathrm{p}_B$ that May-Sigurdsson must resort to the qf-model structure on $\mathcal{T}\mathrm{op}_B$. since the generating qf-cofibrations are fiberwise Hurewicz cofibrations, one has enough control over them to prove the glueing lemma. The verification that a relative J_B^Σ-cell complex in $\mathcal{S}\mathrm{p}_B$ is in fact a stable equivalence requires this usage of fiberwise Hurewicz cofibrations. There are further compatibilities between the model structure and cofiber sequences built out of fiberwise cones that are needed. See [1, §5.3–5.6] for a detailed explanation of the benefits obtained from building up from the qf-model structure. We will write $\mathrm{ho}\,\mathcal{S}\mathrm{p}_B$ for the homotopy category of spectra over B associated to the stable model category of spectra over B.

The base-change functors on ex-spaces extend to give stable base-change functors

$$\begin{aligned} f_!\colon \mathcal{S}\mathrm{p}_A &\longrightarrow \mathcal{S}\mathrm{p}_B \\ f^*\colon \mathcal{S}\mathrm{p}_B &\longrightarrow \mathcal{S}\mathrm{p}_A \qquad \text{that satisfy } f_! \dashv f^* \dashv f_*. \\ f_*\colon \mathcal{S}\mathrm{p}_A &\longrightarrow \mathcal{S}\mathrm{p}_B \end{aligned}$$

They are defined levelwise. In order to derive homotopically meaningful statements, we should work with the derived versions of the base-change functors, by applying an appropriate cofibrant or fibrant approximation before applying the functor. From now on, we will implicitly mean the derived base-change functor when we write $f_!$, f^* or f_*. Notice that the derived fiber functors

$$i_b^*(-) = (-)_b\colon\ \mathrm{ho}\,\mathcal{S}\mathrm{p}_B \longrightarrow \mathrm{ho}\,\mathcal{S}\mathrm{p}$$

jointly detect the stable equivalences of parametrized spectra.

There is some subtletly with the definition of the base change functor f_*. This stems from the fact that $(f_!, f^*)$ is a Quillen adjunction but (f^*, f_*) is not Quillen, and in fact cannot be made Quillen simlutaneously with $(f_!, f^*)$ for any model structure on $\mathcal{S}\mathrm{p}_B$. (see Counterexample 0.0.1 in [1]).

Example. Recall that $r\colon B \longrightarrow *$ is the projection map to a point. The base-change functor $r_!\colon \mathcal{S}\mathrm{p}_B \longrightarrow \mathcal{S}\mathrm{p}$ has the effect of collapsing all sections to a single basepoint. In particular, the base change $r_!\Sigma_B^\infty(Y,p)_+$ of a fiberwise suspension spectrum to a point is canonically equivalent to the suspension spectrum $\Sigma^\infty Y_+$.

Let (Y,p) be a space over B and suppose given a map $f\colon A \longrightarrow B$. Then the pullback $f^*\Sigma_B^\infty(Y,p)_+$ is canonically equivalent to the suspension spectrum $\Sigma_A^\infty(Z,q)_+$ of the homotopy pullback Z of Y along f. We may explicitly describe the space Z by replacing p by a homotopy equivalent fibration $p'\colon Y' \longrightarrow B$ and then taking the pullback:

$$\begin{array}{ccc} Z & \longrightarrow & Y' \\ {\scriptstyle q}\big\downarrow & & \big\downarrow{\scriptstyle p'} \\ A & \xrightarrow{\ f\ } & B. \end{array}$$

Example. Let $\Delta\colon B \longrightarrow B\times B$ denote the diagonal map. Then there is a canonical equivalence

$$\Delta^*\Delta_! S_B \cong \Delta^*\Sigma_{B\times B}^\infty(B,\Delta)_+ \cong \Sigma_B^\infty(LB, e_0)_+,$$

where $LB = \mathrm{Map}(S^1, B)$ is the free loop space of B and e_0 is the evaluation of a loop at the basepoint $0 \in S^1$. Under this identification, the unit $\eta\colon S_B \longrightarrow \Delta^*\Delta_! S_B$ of the adjunction $(\Delta_!, \Delta^*)$ is the map of suspension spectra over B induced by the inclusion of constant loops $c\colon B \longrightarrow LB$.

Example. There is a projection formula

$$f_!(f^*X \wedge_A Y) \cong X \wedge_B f_! Y$$

relating the pushforward and pullback functors along $f\colon A \longrightarrow B$.

Example. Suppose given a commutative diagram of maps of topological spaces

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle j} \\ C & \xrightarrow{g} & D \end{array}$$

and a spectrum X over C. Then there is a natural map $\alpha\colon f_! i^* X \longrightarrow j^* g_! X$ of spectra over B defined by the composite

$$\alpha\colon f_! i^* \xrightarrow{\eta} f_! i^* g^* g_! \cong f_! f^* j^* g_! \xrightarrow{\epsilon} j^* g_!$$

of the unit and counit for the adjunctions of base change functors induced by f and g. When the square is a homotopy pullback square, then the transformation α is an equivalence of derived functors, often called the Beck-Chevalley isomorphism.

Example. A parametrized spectrum E over B gives rise to a twisted homology and twisted cohomology theory on $\mathcal{T}\mathrm{op}/B$. The values of the theory are defined by the pushforward functor and the global sections functor:

$$E_*(X) = \pi_* r_!(\Sigma^\infty_B X_+ \wedge_B E) \qquad E^*(X) = \pi_{-*} r_* F_B(\Sigma^\infty_B X_+, E).$$

Here $F_B(-,-)$ is the internal hom in parametrized spectra, defined so that $F_B(X,-)$ is right adjoint to $- \wedge_B X$. There is a version of Brown representability in this context which states that any twisted cohomology theory on spaces over B satisfying a version of the Eilenberg-Steenrod axioms is of this form.

The discrepancy between $r_!$ and r_* in the definition of homology and cohomology means that fiberwise duality, as discussed in the next section, does not induce a duality between twisted homology and cohomology. The right framework for Poincaré duality in parametrized homology and cohomology is a more general type of bicategorical duality [1, §19].

4. Fiberwise duality

We say that $X \in \mathrm{ho}\,\mathcal{S}\mathrm{p}_B$ is fiberwise dualizable if it is a dualizable object of the symmetric monoidal category $(\mathcal{S}\mathrm{p}_B, \wedge_B S_B)$. In other words, there exists a parametrized spectrum $Y = DX \in \mathrm{ho}\,\mathcal{S}\mathrm{p}_B$ and coevaluation and evaluation morphisms

$$\mathrm{coev}\colon S_B \longrightarrow X \wedge_B Y \qquad \mathrm{eval}\colon Y \wedge_B X \longrightarrow S_B$$

such that

$$X \cong S_B \wedge_B X \xrightarrow{\mathrm{coev} \wedge 1} X \wedge_B Y \wedge_B X \xrightarrow{1 \wedge \mathrm{eval}} X \wedge_B S_B \cong X$$

$$Y \cong Y \wedge_B S_B \xrightarrow{1 \wedge \mathrm{coev}} Y \wedge_B X \wedge_B Y \xrightarrow{\mathrm{eval} \wedge 1} S_B \wedge_B Y \cong Y$$

are homotopic to the identity maps on X and Y.

The pullback functor f^* is strong monoidal, so it follows that the derived fiber spectra (X_b, Y_b) form a dual pair in $\mathcal{S}\mathrm{p}$. In fact, a parametrized spectrum X is fiberwise dualizable if and only of the canonical map

$$F_B(X, S_B) \wedge_B X \longrightarrow F_B(X, X)$$

is a stable equivalence over B. But since the derived fiber functors jointly detect stable equivalences, this is equivalent to the condition that

$$F(X_b, S) \wedge X \longrightarrow F(X_b, X_b)$$

is a stable equivalence of spectra for every $b \in B$. Thus we see that X is fiberwise dualizable if and only if each X_b is Spanier-Whitehead dualizable as a spectrum, i.e. is a finite cell spectrum.

Example. Suppose that $f\colon E \longrightarrow B$ is a fibration and each fiber E_b is a homotopy finitely dominated space, i.e. is a retract of an up-to-homotopy finite CW complex. Then the fiberwise suspension spectrum $\Sigma^\infty_B(E, f)_+$ has fiber

$$(\Sigma^\infty_B(E, f)_+)_b \cong \Sigma^\infty_+ E_b,$$

a finite cell spectrum. Therefore, $f_! S_E = \Sigma^\infty_B(E, f)_+$ is fiberwise dualizable over B.

Example. As a special case of the previous example, suppose that $f\colon E \longrightarrow B$ is a fiber bundle of smooth manifolds with compact fiber M. Then we can describe the dual of $f_! S_E$ explicitly. Let $TM = \ker(TE \longrightarrow f^*TB)$ be the vertical tangent bundle. As a virtual vector bundle on E, we may write $TM = TE - f^*TB$. Write S_E^{TM} for the sphere bundle over E given by fiberwise one point compactification of TM. Write S_E^{-TM} for its inverse under $\wedge_E$. This is the spherical fibration over E associated to the virtual vector space $-TM$.

I claim that $(f_! S_E, f_! S_E^{-TM})$ is a dual pair in $\mathrm{ho}\,\mathcal{S}\mathrm{p}_B$. This is fiberwise Atiyah duality for M, parametrized over B. We can construct the coevaluation and evaluation morphisms explicitly. Let $j\colon E \longrightarrow B \times \mathbf{R}^n$ be an embedding that agrees with f on the first factor. Then there is a tubular neighborhood of E in $B \times \mathbf{R}^n$ which is homeomorphic to the sphere bundle $p\colon S_E^{\nu(j)} \longrightarrow E$ associated to the normal bundle of the embedding j. The fiberwise Pontjagin-Thom collapse map

$$\mathrm{PT}\colon S^n_B = B \times S^n \longrightarrow S_E^{\nu(j)}/B = f_! S_E^{\nu(j)}$$

takes values in the Thom space obtained by collapsing the section of $S_E^{\nu(j)}$ at ∞ to a copy of B. This is implemented by the base-change functor $f_!$. We compose the fiberwise Pontrjagin Thom collapse map with the Thom diagonal $v \mapsto p(v) \wedge v$:

$$S^n_B \xrightarrow{\mathrm{PT}} f_! S_E^{\nu(j)} \xrightarrow{\Delta} E_+ \wedge_B f_! S_E^{\nu(j)}.$$

Since $\nu(j) \oplus TE \cong f^*TB \oplus \epsilon^n$, there is an equality of virtual vector bundles $\nu(j) - \epsilon^n = f^*Tb - TE = -TM$. Apply fiberwise desuspension to the composite of the fiberwise Pontrjagin-Thom collapse map and the Thom diagonal gives the coevaluation morphism for the duality:

$$\mathrm{coev}\colon S_B \xrightarrow{\Sigma_B^{-n}\mathrm{PT}} f_! S_E^{-TM} \xrightarrow{\Sigma_B^{-n}\Delta} f_! S_E \wedge_B f_! S_E^{-TM}.$$

To construct the evaluation morphism, consider the composite of embeddings

$$E \xrightarrow{\Delta} E \times_B E \xrightarrow{j \times 1} (B \times \mathbf{R}^n) \times_B E \cong E \times \mathbf{R}^n.$$

Since the composite is isotopic to the zero-section, the normal bundle $\nu((j \times 1) \circ \Delta)$ is homeomorphic to the trivial bundle ϵ^n on E. The fiberwise Pontrjagin-Thom collapse map for Δ relative to the ambient space $E \times \mathbf{R}^n$ takes the form

$$\mathrm{PT}\colon (f \times f)_! S_{E \times_B E}^{\nu(j \times 1)} \longrightarrow f_! S^{\nu}((j \times 1) \circ \Delta)_E \cong f_! S^n_E.$$

Desuspending by the fiberwise n-sphere, and composing with the projection to B defines the evaluation morphism

$$\mathrm{eval}\colon f_! S_E^{-TM} \wedge_B f_! S_E \cong (f \times f)_! S_{E \times_B E}^{\nu(j \times 1)} \xrightarrow{\Sigma_B^{-n} \mathrm{PT}} f_! S_E = \Sigma_B^\infty (E, f)_+ \xrightarrow{f} S_B$$

I have borrowed a few ideas and notation from Rezk's nice treatment of Atiyah duality in the non-parametrized setting [2].

References

[1] J.P. May and J. Sigurdsson, *Parametrized homotopy theory*, Mathematical Surveys and Monographs, vol. 132, American Mathematical Society, 2006.

[2] C. Rezk, *Frobenius pairs and Atiyah duality*, arXiv:1303.3567.