The algebra of the spheres

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The n-sphere is the topological space

$$S^n = \{(x_0, \dots, x_n) \in \mathbf{R}^{n+1} \colon \sum_i x_i^2 = 1\}.$$

For example, S^1 is the unit circle inside of $\mathbf{R}^2 \cong \mathbf{C}$.
A map $\gamma \colon S^1 \longrightarrow S^1$ with $\gamma(1) = 1$ determines a loop in S^1.

$$\begin{aligned} \text{Define } \Omega S^1 &= \mathrm{Map}(S^1, S^1) \\ &= \text{the space of loops in } S^1. \end{aligned}$$

Identify loops if they are homotopic:

$$\gamma \simeq \eta \quad \iff \quad \begin{aligned} &\exists H \colon S^1 \times [0,1] \longrightarrow S^1 \\ &H(x,0) = \gamma(x) \\ &H(x,1) = \eta(x) \end{aligned}$$

$$\begin{aligned} \text{Define } \pi_1 S^1 &= \Omega S^1 / \simeq \\ &= \text{the set of homotopy classes of loops in } S^1. \end{aligned}$$

The composition of paths makes $\pi_1 S^1$ into a group.

Theorem (ancient wisdom of human consciousness...)

$\pi_1 S^1 \cong \mathbb{Z}$

Proof.

Covering space theory! Every loop $\gamma\colon S^1 \longrightarrow S^1$ lifts uniquely to a path $\overline{\gamma}\colon [0,1] \longrightarrow \mathbb{R}$

$$\begin{array}{ccc} [0,1] & \overset{\overline{\gamma}}{\dashrightarrow} & \mathbb{R} \\ {\scriptstyle \exp}\downarrow & & \downarrow{\scriptstyle \exp} \\ S^1 & \overset{\gamma}{\longrightarrow} & S^1 \end{array}$$

and then $[\gamma] \longmapsto \overline{\gamma}(1) \in \mathbb{Z}$ is the isomorphism intuited by our ancestors. $\square$

Define $\Omega^n S^k = \mathrm{Map}(S^n, S^k)$

$= \text{the space of maps } S^n \longrightarrow S^k.$

Given $\gamma, \eta \in \Omega^n S^k$, define $\gamma + \eta$ by

$$S^n \xrightarrow{\text{pinch}} S^n \vee S^n \xrightarrow{\gamma \vee \eta} S^k.$$

Mild annoyance: $+$ depends on where we pinch S^n.

- $\Omega^n S^k$ isn't quite a group, only a homotopical group.

Solution: identify maps if they are homotopic.

Define $\pi_n S^k = \Omega^n S^k / \simeq$

$= \text{the set of homotopy classes of maps } S^n \longrightarrow S^k.$

Then $\pi_n S^k$ is a group. ☺
In fact, an abelian group when $n > 1$. ☺☺☺

What is $\pi_2 S^2$? Given $\gamma\colon S^2 \longrightarrow S^2$, define

$$\mathrm{degree}(\gamma) = |\gamma^{-1}\{x\}|_{\mathrm{mult}} \quad x \in S^2 \text{ a generic value.}$$

Then $[\gamma] \longmapsto \mathrm{degree}(\gamma)$ defines an isomorphism $\pi_2 S^2 \cong \mathbb{Z}$, and similarly $\pi_n S^n \cong \mathbb{Z}$.

What about $\pi_1 S^2$? After perturbation, a loop $\gamma\colon S^1 \longrightarrow S^2$ misses the north pole $\infty \in S^2 \cong \mathbb{C} \cup \{\infty\}$. Then the contraction

$$H\colon S^2 \setminus \{\infty\} \times [0,1] \longrightarrow S^2 \setminus \{\infty\}$$
$$H(z,t) = (1-t)z$$

provides a nullhomotopy $\gamma \simeq 0$. Therefore, $\pi_1 S^2 = 0$.

In fact, $\pi_n S^k = 0$ whenever $n < k$.

Let's record our progress:

	S^1	S^2	S^3	S^4	S^5	S^6	S^7	S^8	S^9	S^{10}	S^{11}
π_1	$\mathbb{Z}$	0	0	0	0	0	0	0	0	0	0
π_2		$\mathbb{Z}$	0	0	0	0	0	0	0	0	0
π_3			$\mathbb{Z}$	0	0	0	0	0	0	0	0
π_4				$\mathbb{Z}$	0	0	0	0	0	0	0
π_5					$\mathbb{Z}$	0	0	0	0	0	0
π_6						$\mathbb{Z}$	0	0	0	0	0
π_7							$\mathbb{Z}$	0	0	0	0
π_8								$\mathbb{Z}$	0	0	0
π_9									$\mathbb{Z}$	0	0
π_{10}										$\mathbb{Z}$	0
π_{11}											$\mathbb{Z}$

Let's record our progress:

	S^1	S^2	S^3	S^4	S^5	S^6	S^7	S^8	S^9	S^{10}	S^{11}
π_1	$\mathbb{Z}$	0	0	0	0	0	0	0	0	0	0
π_2	0	$\mathbb{Z}$	0	0	0	0	0	0	0	0	0
π_3	0		$\mathbb{Z}$	0	0	0	0	0	0	0	0
π_4	0			$\mathbb{Z}$	0	0	0	0	0	0	0
π_5	0				$\mathbb{Z}$	0	0	0	0	0	0
π_6	0					$\mathbb{Z}$	0	0	0	0	0
π_7	0						$\mathbb{Z}$	0	0	0	0
π_8	0							$\mathbb{Z}$	0	0	0
π_9	0								$\mathbb{Z}$	0	0
π_{10}	0									$\mathbb{Z}$	0
π_{11}	0										$\mathbb{Z}$

In fact, $\pi_n S^1 = 0$ for $n > 1$.
The next case to consider is $\pi_3 S^2$.

The Hopf fibration $\eta\colon S^3 \longrightarrow S^2$ is defined by the composite

$$\begin{aligned} S^3 \subset \mathbb{R}^4 \cong \mathbb{C}^2 &\xrightarrow{\text{quotient}} \mathbb{C} \cup \{\infty\} \cong S^2 \\ (z, w) &\longmapsto z/w \end{aligned}$$

In fact, $\pi_3 S^2 = \mathbb{Z}\{\eta\}$.

A similar technique using quaternions and octonions defines

$$\nu\colon S^7 \longrightarrow S^4 \quad \text{and} \quad \sigma\colon S^{15} \longrightarrow S^8$$

and it turns out that

$$\pi_7 S^4 = \mathbb{Z}\{\nu\} \oplus \mathbb{Z}/12 \quad \text{and} \quad \pi_{15} S^8 \cong \mathbb{Z}\{\sigma\} \oplus \mathbb{Z}/120.$$

[Hopf, 1935]

$\pi_{n+k}S^k$	$k=1$	2	3	4	5	6	7	8	9	10	11
$n+k=1$	$\mathbb{Z}$	0	0	0	0	0	0	0	0	0	0
2	0	$\mathbb{Z}$	0	0	0	0	0	0	0	0	0
3	0	$\mathbb{Z}$	$\mathbb{Z}$	0	0	0	0	0	0	0	0
4	0			$\mathbb{Z}$	0	0	0	0	0	0	0
5	0				$\mathbb{Z}$	0	0	0	0	0	0
6	0					$\mathbb{Z}$	0	0	0	0	0
7	0			$\mathbb{Z}\oplus\mathbb{Z}/12$			$\mathbb{Z}$	0	0	0	0
8	0							$\mathbb{Z}$	0	0	0
9	0								$\mathbb{Z}$	0	0
10	0									$\mathbb{Z}$	0
11	0										$\mathbb{Z}$

Maybe the diagonal pattern continues: $\pi_4 S^3 = \mathbb{Z}$?

$\pi_{n+k}S^k$	$k=1$	2	3	4	5	6	7
$n+k=1$	$\mathbb{Z}$	0	0	0	0	0	0
2	0	$\mathbb{Z}$	0	0	0	0	0
3	0	$\mathbb{Z}$	$\mathbb{Z}$	0	0	0	0
4	0		$\mathbb{Z}/2$	$\mathbb{Z}$	0	0	0
5	0	$\mathbb{Z}/2$		$\mathbb{Z}/2$	$\mathbb{Z}$	0	0
6	0		$\mathbb{Z}/12$		$\mathbb{Z}/2$	$\mathbb{Z}$	0
7	0			$\mathbb{Z}\oplus\mathbb{Z}/12$		$\mathbb{Z}/2$	$\mathbb{Z}$
8	0				$\mathbb{Z}/24$		$\mathbb{Z}/2$
9	0					$\mathbb{Z}/24$	
10	0						$\mathbb{Z}/24$
11	0						

$\pi_{n+k}S^k$	$k=1$	2	3	4	5	6	7	
$n+k=1$	$\mathbb{Z}$	0	0	0	0	0	0	
2	0	$\mathbb{Z}$	0	0	0	0	0	
3	0	$\mathbb{Z}$	$\mathbb{Z}$	0	0	0	0	
4	0	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}$	0	0	0	
5	0	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}$	0	0	
6	0		$\mathbb{Z}/12$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}$	0	
7	0			$\mathbb{Z} \oplus \mathbb{Z}/12$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}$	
8	0				$\mathbb{Z}/24$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	
9	0					$\mathbb{Z}/24$	$\mathbb{Z}/2$	$\mathbb{Z}$
10	0						$\mathbb{Z}/24$	$\mathbb{Z}$
11	0							$\mathbb{Z}$

$\pi_{n+k}S^k$	$k=1$	2	3	4	5	6	7
$n+k=1$	$\mathbb{Z}$	0	0	0	0	0	0
2	0	$\mathbb{Z}$	0	0	0	0	0
3	0	$\mathbb{Z}$	$\mathbb{Z}$	0	0	0	0
4	0	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}$	0	0	0
5	0	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}$	0	0
6	0	$\mathbb{Z}/12$	$\mathbb{Z}/12$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}$	0
7	0		$\mathbb{Z}/2$	$\mathbb{Z}\oplus\mathbb{Z}/12$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}$
8	0			$(\mathbb{Z}/2)^2$	$\mathbb{Z}/24$	$\mathbb{Z}/2$	$\mathbb{Z}/2$
9	0				$\mathbb{Z}/2$	$\mathbb{Z}/24$	$\mathbb{Z}/2$
10	0					0	$\mathbb{Z}/24$
11	0						0

$\pi_{n+k}S^k$	$k=1$	2	3	4	5	6	7
$n+k=1$	$\mathbb{Z}$	0	0	0	0	0	0
2	0	$\mathbb{Z}$	0	0	0	0	0
3	0	$\mathbb{Z}$	$\mathbb{Z}$	0	0	0	0
4	0	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}$	0	0	0
5	0	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}$	0	0
6	0	$\mathbb{Z}/12$	$\mathbb{Z}/12$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}$	0
7	0	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z} \oplus \mathbb{Z}/12$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}$
8	0		$\mathbb{Z}/2$	$(\mathbb{Z}/2)^2$	$\mathbb{Z}/24$	$\mathbb{Z}/2$	$\mathbb{Z}/2$
9	0			$(\mathbb{Z}/2)^2$	$\mathbb{Z}/2$	$\mathbb{Z}/24$	$\mathbb{Z}/2$
10	0				$\mathbb{Z}/2$	0	$\mathbb{Z}/24$
11	0					$\mathbb{Z}$	0

$\pi_{n+k}S^k$	$k=1$	2	3	4	5	6	7
$n+k=1$	$\mathbb{Z}$	0	0	0	0	0	0
2	0	$\mathbb{Z}$	0	0	0	0	0
3	0	$\mathbb{Z}$	$\mathbb{Z}$	0	0	0	0
4	0	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}$	0	0	0
5	0	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}$	0	0
6	0	$\mathbb{Z}/12$	$\mathbb{Z}/12$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}$	0
7	0	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}\oplus\mathbb{Z}/12$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}$
8	0	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$(\mathbb{Z}/2)^2$	$\mathbb{Z}/24$	$\mathbb{Z}/2$	$\mathbb{Z}/2$
9	0	$\mathbb{Z}/3$	$\mathbb{Z}/3$	$(\mathbb{Z}/2)^2$	$\mathbb{Z}/2$	$\mathbb{Z}/24$	$\mathbb{Z}/2$
10	0	$\mathbb{Z}/15$	$\mathbb{Z}/15$	$\mathbb{Z}/3\oplus\mathbb{Z}/24$	$\mathbb{Z}/2$	0	$\mathbb{Z}/24$
11	0	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/15$	$\mathbb{Z}/2$	$\mathbb{Z}$	0

$\pi_{n+k}S^k$ stabilizes for $k > n+1$. [Freudenthal, 1937]

Denote the common value by $\lim\limits_{k\to\infty} \pi_{n+k}S^k = \pi_n S$.

$$\pi_0 S = \mathbb{Z},\ \ \pi_1 S = \mathbb{Z}/2,\ \ \pi_2 S = \mathbb{Z}/2,\ \ \pi_3 S = \mathbb{Z}/24,\ \ \pi_4 S = 0 \ldots$$

The groups $\pi_n S$ are called the stable homotopy groups of spheres:

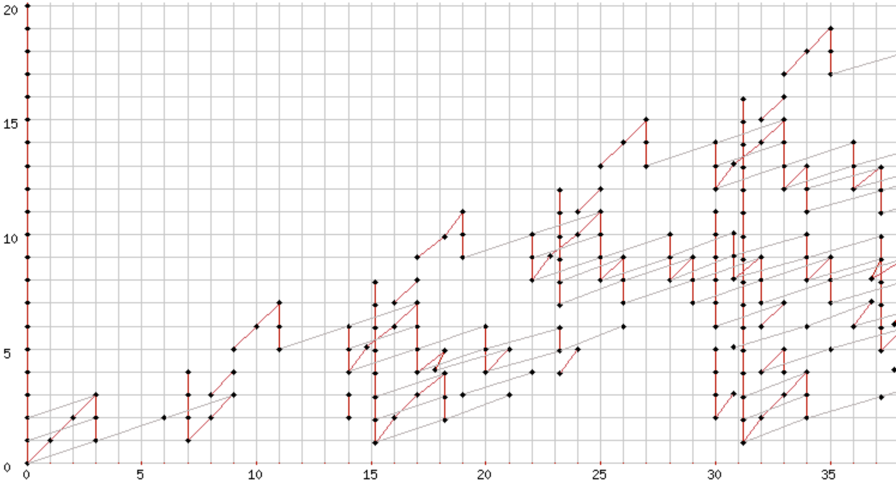
$$\pi_0 S = \mathbb{Z}, \ \ \pi_1 S = \mathbb{Z}/2, \ \ \pi_2 S = \mathbb{Z}/2, \ \ \pi_3 S = \mathbb{Z}/24, \ \ \pi_4 S = 0,$$
$$\pi_5 S = 0, \ \ \pi_6 S = \mathbb{Z}/2, \ \ \pi_7 S = \mathbb{Z}/240, \ \ \pi_8 S = (\mathbb{Z}/2)^2, \ \ \pi_9 S = (\mathbb{Z}/2)^3$$
$$\pi_{10} = \mathbb{Z}/6, \ \ \pi_{11} S = \mathbb{Z}/504, \ \ \pi_{12} S = 0, \ \ \pi_{13} S = \mathbb{Z}/3, \ldots$$

- $\pi_n S$ is a finite abelian group for $n > 0$ [Serre, 1950].
- There is a multiplication

  $$\pi_m S^j \times \pi_n S^k \longrightarrow \pi_{m+n} S^{j+k}$$

  making $\pi_* S$ into a graded ring.
- In positive degrees, the ring $\pi_* S$ is nilpotent ($x^k = 0$ for $k \gg 0$) [Nishida, 1973].
- We can think of the projection $\pi_* S \longrightarrow \pi_0 S = \mathbb{Z}$ as a fattening of the integers by nilpotent elements.

The Adams Spectral Sequence for $p = 2$



$$\mathrm{Ext}_{\mathcal{A}}^{s,t}(\mathbf{F}_2, \mathbf{F}_2) \Longrightarrow \pi_{t-s} S_{(2)}$$

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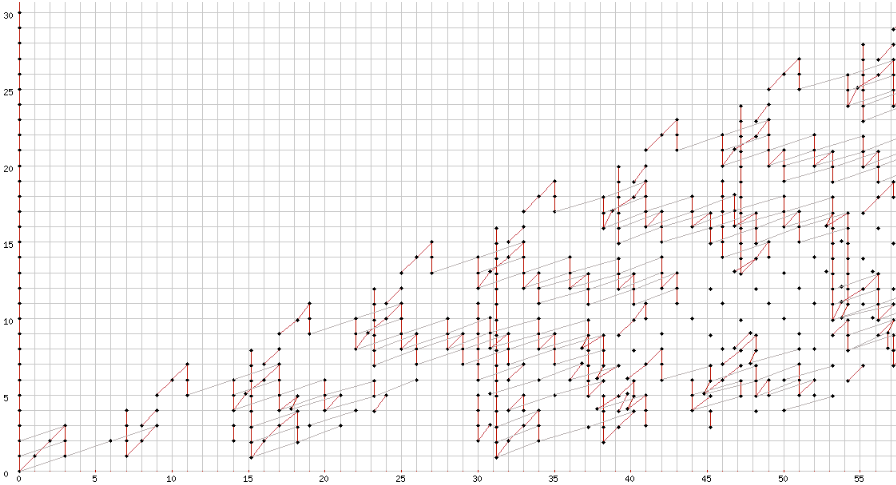

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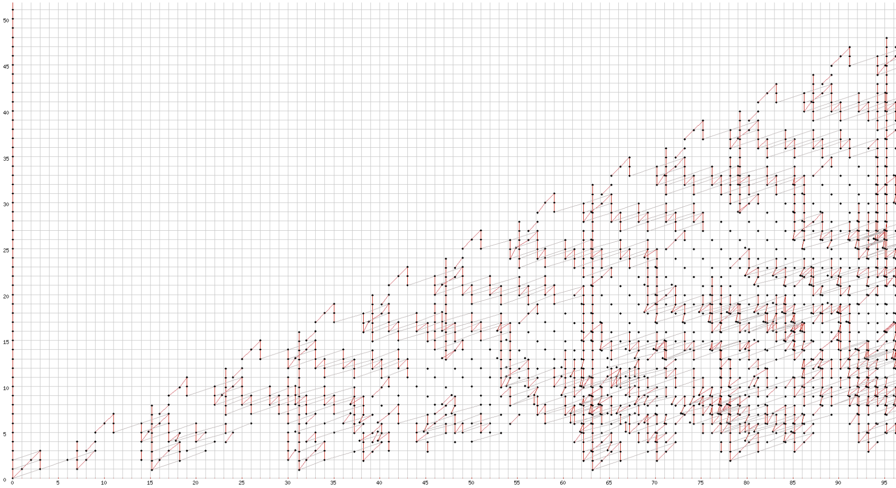

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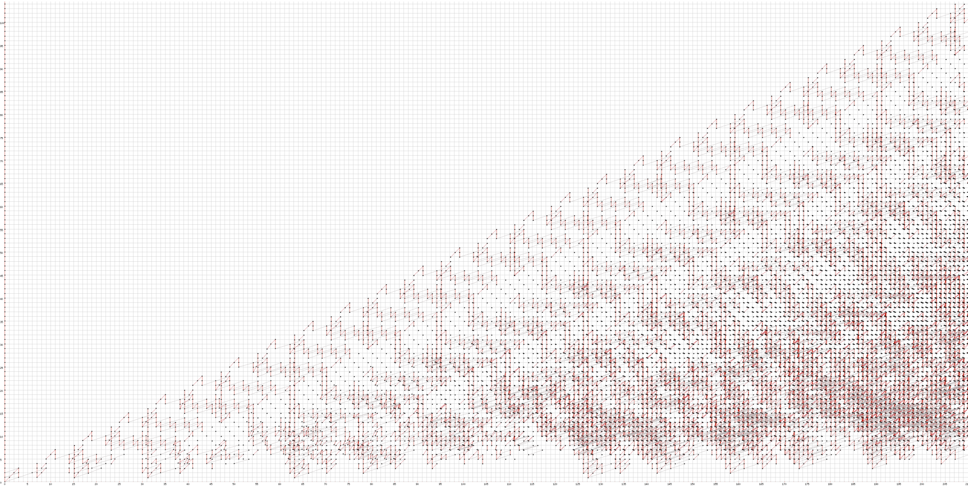

$$\mathrm{Ext}_{\mathcal{A}}^{s,t}(\mathbf{F}_2, \mathbf{F}_2) \Longrightarrow \pi_{t-s} S_{(2)}$$

The pattern along the top diagonal of the Adams spectral sequence detects an infinite family of elements in the stable homotopy groups called the image of J:

Theorem (Adams, Quillen, Sullivan 1965–1971)

- *$\pi_{8k+1}S$ and $\pi_{8k+2}S$ contain a summand isomorphic to $\mathbb{Z}/2$.*
- *$\pi_{4k}S$ contains a summand isomorphic to $\mathbb{Z}/d$, where*

$$d = \text{denominator of } B_{2k}/4k.$$

Here, the Bernoulli numbers B_n are defined by

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} \frac{B_n x^n}{n!}$$

and are related to zeta values:

$$\zeta(1-2k) = (-1)^k \frac{B_{2k}}{2k}.$$

$$\Omega^n S^n = \mathrm{Map}(S^n, S^n)$$
$$= \text{the space of maps } S^n \longrightarrow S^n.$$

The algebraic stabilization $\pi_n S = \lim_{k\to\infty} \pi_{n+k} S^k$ has a topological avatar:

$$\Omega^\infty S^\infty := \bigcup_{k\geq 0} \Omega^k S^k.$$

The $+$ and $\cdot$ on $\pi_* S$ come from topological operations making $\Omega^\infty S^\infty$ into a homotopical ring.

Since $\pi_n \Omega^k S^k = \mathrm{Map}(S^{n+k}, S^k)/\simeq$,

$$\pi_n \Omega^\infty S^\infty = \lim_{k\to\infty} \pi_n \Omega^k S^k = \lim_{k\to\infty} \pi_{n+k} S^k = \pi_n S.$$

This suggests that the "universal sphere" is $S = \Omega^\infty S^\infty$.

For each group G, there is a unique (up to $\simeq$) space BG satisfying

$$\pi_n BG = \begin{cases} G & \text{if } n = 1, \\ 0 & \text{if } n \neq 1. \end{cases}$$

Let $\Sigma_k = \mathrm{Aut}\{1, \ldots, k\}$. Then disjoint union and cartesian product of sets define maps

$$+\colon B\Sigma_j \times B\Sigma_k \longrightarrow B\Sigma_{j+k}$$
$$\cdot\colon B\Sigma_j \times B\Sigma_k \longrightarrow B\Sigma_{jk}$$

and so:

$$\coprod_{k \geq 0} B\Sigma_k \text{ is a homotopical rig (rig = ring without negatives)}$$

In fact, it is the free homotopical rig on the category of finite sets!

The homomorphism $\mathbb{N} \longrightarrow \mathbb{Z}$ is a group completion:

$\mathbb{Z}$ is obtained from $\mathbb{N}$ by formally adding negatives.

Instead of working only with the cardinality of finite sets, we can take their automorphisms into account and consider Σ_k.

Theorem (Barratt-Priddy-Quillen 1972)

There is a group completion of homotopical rings

$$\coprod_{k \geq 0} B\Sigma_k \longrightarrow \Omega^\infty S^\infty.$$

Equivalently, there is an isomorphism of integral homology groups

$$H_*(\mathbb{Z} \times B\Sigma_\infty) \cong H_*(\Omega^\infty S^\infty).$$

Group completion is black magic on homotopy groups:

$$\pi_n(B\Sigma_k) = 0 \text{ for } n > 1, \text{ but } \pi_n \Omega^\infty S^\infty = \pi_n S!$$

Brave New Algebra

In stable homotopy theory, the group completion $S = \Omega^\infty S^\infty$ is the universal base ring. Just as

$\mathbb{Z}$-algebras = rings,

S-algebras = homotopical rings = setting for derived algebra.

Examples of S-algebras:

- $\mathbb{Z}$-algebras, by restriction along $S \longrightarrow \mathbb{Z}$.
- given a space X, the S-algebra $S[\Omega X]$ is a refinement of the group ring $\mathbb{Z}[\pi_1 X]$
- BO, BU = the classifying spaces for vector bundles over $\mathbb{R}, \mathbb{C}$
- differential graded algebras, in particular those encoding intersections in algebraic geometry.

Brave New Algebra

I study the algebraic properties of S-algebras. An S-algebra R has

- a space of units $R^\times$,
- a topological version of Hochschild homology $THH_*(R)$,
- a notion of algebraic K-theory $K_*(R)$.

These are new invariants, and their study is connected to lots of interesting algebraic geometry, number theory, and differential topology.