CATEGORIES AND FUNCTORS

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Category theory is a way of formalizing the manifestation of analogous patterns in different contexts. The basic idea is that the functions between mathematical objects are as important as the objects themselves. This perspective helps us perceive unity and universality, and gives us a language to efficiently describe rich mathematical structures.

Definition 1.1. A *category* $\mathcal{C}$ consists of

- a collection of objects $X, Y, Z, \ldots$,
- for any pair of objects X and Y, a set $\mathcal{C}(X, Y)$ of *morphisms from* X *to* Y. We write elements $f \in \mathcal{C}(X, Y)$ as $f\colon X \longrightarrow Y$, $g\colon X \longrightarrow Y, \ldots$
- a composition operation

$$\begin{aligned} \circ\colon \mathcal{C}(Y, Z) \times \mathcal{C}(X, Y) &\longrightarrow \mathcal{C}(X, Z) \\ (g, f) &\longmapsto g \circ f \end{aligned}$$

- for each object X in $\mathcal{C}$, a distinguished morphism $\mathrm{id}_X\colon X \longrightarrow X$ called the *identity morphism*,

The morphisms in $\mathcal{C}$ must satisfy the following conditions:

- composition of morphisms is associative: given morphisms $f\colon X \longrightarrow Y$, $g\colon Y \longrightarrow Z$, and $h\colon Z \longrightarrow W$, there is an equality

$$(h \circ g) \circ f = h \circ (g \circ f)$$

  of morphisms from X to W.
- identity morphisms act as identities: for any morphism $f\colon X \longrightarrow Y$,

$$f \circ \mathrm{id}_X = f \qquad \text{and} \qquad \mathrm{id}_Y \circ f = f$$

Given a morphism $f\colon X \longrightarrow Y$ in a category, we call X the *domain*, or *source* of f, and we call Y the *codomain*, or *target* of f.

Examples 1.2.

(1) The category $\mathcal{S}et$ of sets: the objects are sets and a morphism $f\colon X \longrightarrow Y$ from a set X to a set Y is a function. Composition of morphisms is defined by

$$(g \circ f)(x) = g(f(x)), \qquad \text{where } f\colon X \longrightarrow Y, g\colon Y \longrightarrow Z, \text{ and } x \in X,$$

and the identity morphisms $\mathrm{id}_X\colon X \longrightarrow X$ are the identify functions $\mathrm{id}_X(x) = x$.

(2) The category $\mathcal{T}$op of topological spaces: the objects are topological spaces and the morphisms are continuous maps (also known as continuous functions). Composition and the identity morphisms are defined as in $\mathcal{S}$et.

(3) The category $\mathcal{M}$an of topological manifolds: the objects are n-manifolds (for all $n \geq 0$) and the morphisms are continuous maps.

(4) The category $\mathcal{G}$rp of groups: the objects are groups $G, H, \dots$ and the morphisms are group homomorphisms, i.e. a morphism is a function $f \colon G \longrightarrow H$ satisfying

$$f(g \cdot h) = f(g) \cdot f(h) \qquad \text{for all } g, h \in G.$$

(5) The category $\mathcal{A}$b$\mathcal{G}$rp of abelian groups: the objects are abelian groups $A, B, \dots$ and the morphisms are group homomorphisms.

(6) The category $\mathcal{V}\text{ect}_k$ of vector spaces over a field k: the objects are vector spaces $V, W, \dots$ over k and the morphisms are linear maps (also known as linear transformations, linear functions, ...).

(7) The category $\mathcal{M}$etric of metric spaces: the objects are metric spaces (X, d) and the morphisms $f \colon (X, d_X) \longrightarrow (Y, d_Y)$ are linear isometries, meaning functions $f \colon X \longrightarrow Y$ for which

$$d_X(x, y) = d_Y(f(x), f(y)) \qquad \text{for all } x, y \in X.$$

(8) The category $[n]$ has $n+1$ objects: $0, 1, \dots, n$. The morphisms in $[n]$ are specified by:

$$[n](i, j) = \begin{cases} * & \text{if } i \leq j, \\ \varnothing & \text{if } i > j. \end{cases}$$

Here the symbol $*$ is a generic name for a set with a single element. To be completely precise, we could use $* = \{0\}$, for example, but the name of the element of the set $*$ is not important. Unlike the previous examples, we can depict the category $[n]$ quite succinctly:

$$[n] = (0 \longrightarrow 1 \longrightarrow \cdots \longrightarrow n)$$

(9) If $(X, \leq)$ is a partially ordered set, we may view X as a category whose objects are the elements $x \in X$ and whose morphisms are specified by:

$$X(x, y) = \begin{cases} * & \text{if } x \leq y, \\ \varnothing & \text{if } x > y. \end{cases}$$

(10) If $(G, \cdot, e)$ is a group, then we may consider G as a category $\underline{G}$ with a single object $*$ by declaring that the set of morphisms from $*$ to $*$ is $\underline{G}(*, *) = G$. The composition operation $\circ \colon \underline{G} \times \underline{G} \longrightarrow \underline{G}$ is defined to be the group operation $\cdot$ and the identity morphism $\mathrm{id}_* \colon * \longrightarrow *$ is defined to be the identity element $e \in G$.

Exercise 1.3. In examples 1.2.(8) and 1.2.(9), we did not specify the identity morphisms. What are they and why?

Exercise 1.4. We say that a category $\mathcal{C}$ is *small* if the collection of all of the objects of $\mathcal{C}$ is in fact a set. Which of the examples in 1.2 are small categories?

Exercise 1.5. Suppose that $f\colon X \longrightarrow Y$ and $g\colon Y \longrightarrow Y$ are morphisms satisfying $g \circ f = f$. Does it follow that $g = \mathrm{id}_Y$? Prove the statement or provide a counterexample.

Exercise 1.6. Suppose that $f\colon X \longrightarrow Y$ and $g\colon X \longrightarrow X$ are morphisms satisfying $f \circ g = f$. Does it follow that $g = \mathrm{id}_X$? Prove the statement or provide a counterexample.

Definition 1.7. A morphism $f\colon X \longrightarrow Y$ in a category $\mathcal{C}$ is called an *isomorphism* if there exists a morphism $f^{-1}\colon Y \longrightarrow X$ in $\mathcal{C}$ such that

$$f^{-1} \circ f = \mathrm{id}_X \qquad \text{and} \qquad f \circ f^{-1} = \mathrm{id}_Y\,.$$

If every morphism in $\mathcal{C}$ is an isomorphism, then we call $\mathcal{C}$ a *groupoid*.

Exercise 1.8. What are the isomorphisms in $\mathcal{S}\mathrm{et}$? In $\mathcal{G}\mathrm{rp}$? In $\mathcal{T}\mathrm{op}$? In $\mathcal{V}\mathrm{ect}_k$? In $[n]$? In $\underline{G}$? Are any of these categories groupoids?

The mantra of category theory is that morphisms are more fundamental than objects. This begs the question: what is a morphism of categories?

Definition 1.9. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A functor $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ from $\mathcal{C}$ to $\mathcal{D}$ consists of an assignment

$$X \longmapsto F(X)$$

of an object $F(X)$ of $\mathcal{D}$ to each object X of $\mathcal{C}$, as well as a function on morphism sets

$$\mathcal{C}(X,Y) \xrightarrow{F(-)} \mathcal{D}(F(X),F(Y))$$

which respects the composition of morphisms and preserves identity morphisms:

$$F(g \circ f) = F(g) \circ F(f) \qquad \text{and} \qquad F(\mathrm{id}_X) = \mathrm{id}_{F(X)}\,. \tag{1.1}$$

In brief, we say that the conditions in (1.1) make the assignment $F(-)$ *functorial*.

Examples 1.10.

(1) The free abelian group functor $\mathbf{Z}(-)\colon \mathcal{S}\mathrm{et} \longrightarrow \mathcal{A}\mathrm{b}\mathcal{G}\mathrm{rp}$ is defined by taking a set X to the abelian group consisting of finite formal sums of elements of X:

$$\mathbf{Z}X = \{\sum_{x \in X} n_x \cdot x \mid n_x \in \mathbf{Z}, n_x = 0 \text{ for all but finitely many } x \in X\}.$$

The functor $\mathbf{Z}(-)$ is defined on morphisms by taking a function $f\colon X \longrightarrow Y$ of sets to the group homomorphism

$$\begin{aligned} \mathbf{Z}f\colon \mathbf{Z}X &\longrightarrow \mathbf{Z}Y \\ \sum_{x\in X} n_x \cdot x &\longmapsto \sum_{x\in X} n_x \cdot f(x) \end{aligned}$$

It is straightforward to check that this assignment is functorial.

(2) The forgetful functor $U\colon \mathcal{A}b\mathcal{G}rp \longrightarrow \mathcal{S}et$ takes an abelian group $(A, +, 0)$ to the underlying set A and takes a group homomorphism to the underlying function of sets.

(3) There is a functor $\mathcal{M}etric \longrightarrow \mathcal{T}op$ that takes a metric space (X, d) to the topological space $(X, \mathcal{T}^d)$ generated by the metric d.

(4) The n-fold product functor $\overbrace{\mathcal{T}op \times \cdots \times \mathcal{T}op}^{n} \longrightarrow \mathcal{T}op$ takes an n-tuple $(X_1, \cdots, X_n)$ of topological spaces to the product topological space $X_1 \times \cdots \times X_n$, and similarly on morphisms.

(5) Suppose that $\mathcal{C}$ is a category. A functor $X\colon [n] \longrightarrow \mathcal{C}$ from the category $[n]$ to $\mathcal{C}$ consists of a list of $n+1$ objects $X_0, \ldots, X_n$ of $\mathcal{C}$ and n composable morphisms between them:

$$X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} X_2 \xrightarrow{f_3} \cdots \xrightarrow{f_n} X_n.$$

Exercise 1.11. Suppose that $\mathcal{C}$ is a category and that G is a group. Show that a functor $F\colon \underline{G} \longrightarrow \mathcal{C}$ from the category $\underline{G}$ with a single object and G as its morphism set to the category $\mathcal{C}$ is equivalent to the data of

- an object X of $\mathcal{C}$, and
- an isomorphism $f_g\colon X \longrightarrow X$ from X to itself for each group element $g \in G$

satisfying the conditions:

$$\begin{aligned} f_g \circ f_h &= f_{gh} \qquad \text{for every } g, h \in G, \text{ and} \\ f_e &= \mathrm{id}_X\,. \end{aligned}$$

We say that F defines an *action* of the group G on X. For example, if $\mathcal{C}$ is the category $\mathcal{M}an$ of manifolds, we have just defined the notion of a group acting on a manifold. If $\mathcal{C}$ is the category $\mathcal{V}ect_k$ of vector spaces, we have just defined the notion of a group acting on a vector space, also known as a *group representation*.

Definition 1.12. Given a functor $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ and a functor $G\colon \mathcal{D} \longrightarrow \mathcal{C}$ in the reverse direction, we say that the pair (F, G) is an *adjunction* (or that (F, G) is an *adjoint pair of functors*) if there are bijections

$$\mathcal{D}(F(X), Y) \cong \mathcal{C}(X, G(Y))$$

for every object X of $\mathcal{C}$ and every object Y of $\mathcal{D}$. These bijections must be suitably compatible for different choices of X and Y, in a sense that we will make precise later. When (F, G) is an adjunction, we say that F is the left adjoint of G and that G is the right adjoint of F.

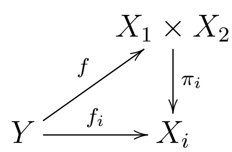
Exercise 1.13. Explain why the functors $\mathbf{Z}(-)$ and U from 1.10.(1) and 1.10.(2) form an adjoint pair $(\mathbf{Z}(-), U)$.

Exercise 1.14. In analogy with 1.10.(2), there is a forgetful functor $U\colon \mathcal{T}\mathrm{op} \longrightarrow \mathcal{S}\mathrm{et}$ that takes a topological space $(X, \mathcal{T})$ to the underlying set X and takes a continuous map to the underlying function of sets. What is the left adjoint of U? What is the right adjoint of U?

Definition 1.15. Let $\mathcal{C}$ be a category and suppose that X_1 and X_2 are objects of $\mathcal{C}$. The *product* of X_1 and X_2 consists of

- an object $X_1 \times X_2$ of $\mathcal{C}$, along with
- morphisms $\pi_1\colon X_1 \times X_2 \longrightarrow X_1$ and $\pi_2\colon X_1 \times X_2 \longrightarrow X_2$

satisfying the following universal characterization: if we are given morphisms $f_1\colon Y \longrightarrow X_1$ and $f_2\colon Y \longrightarrow X_2$ in $\mathcal{C}$ from another object Y of $\mathcal{C}$ to both X_1 and X_2, then there is a unique morphism $f\colon Y \longrightarrow X_1 \times X_2$ making the following diagram commute (for both $i = 1$ and $i = 2$):



Often, we write (f_1, f_2) for the morphism f.

The definition does not assert that the product of X_1 and X_2 exists in $\mathcal{C}$. In general, products may or may not exist in a given category. The point is that the product, if it exists, is specified by its universal characterization.

Exercise 1.16. Given a category $\mathcal{C}$, the product category $\mathcal{C}^2 = \mathcal{C} \times \mathcal{C}$ has as objects pairs (X_1, X_2) of objects of $\mathcal{C}$, and as morphisms pairs (f_1, f_2) of morphisms in $\mathcal{C}$. Note that there is a functor $\Delta\colon \mathcal{C} \longrightarrow \mathcal{C}^2$ taking an object X to the object (X, X) and similarly on morphisms. Explain how we may paraphrase the universal characterization of the product $X_1 \times X_2$ as an adjunction

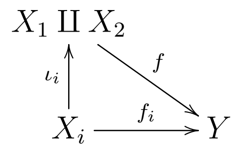
$$\mathcal{C}^2(\Delta Y, (X_1, X_2)) \cong \mathcal{C}(Y, X_1 \times X_2)$$

between the functor Δ and $(-) \times (-)\colon \mathcal{C}^2 \longrightarrow \mathcal{C}$.

Definition 1.17. Let $\mathcal{C}$ be a category and suppose that X_1 and X_2 are objects of $\mathcal{C}$. The *coproduct* of X_1 and X_2 consists of

- an object $X_1 \amalg X_2$ of $\mathcal{C}$, along with
- morphisms $\iota_1 \colon X_1 \longrightarrow X_1 \amalg X_2$ and $\iota_2 \colon X_2 \longrightarrow X_1 \amalg X_2$

satisfying the following universal characterization: if we are given morphisms $f_1 \colon X_1 \longrightarrow Y$ and $f_2 \colon X_2 \longrightarrow Y$ in $\mathcal{C}$ to another object Y of $\mathcal{C}$ from both X_1 and X_2, then there is a unique morphism $f \colon X_1 \amalg X_2 \longrightarrow Y$ making the following diagram commute (for both $i = 1$ and $i = 2$):



Just as for products, the coproduct of a pair of objects may or may not exist in a given category. The definitions specifies what it means to be a coproduct, if such a thing exists.

Exercise 1.18. Explain how we may express the universal characterization of the coproduct $X_1 \amalg X_2$ as an adjunction

$$\mathcal{C}(X_1 \amalg X_2, Y) \cong \mathcal{C}^2((X_1, X_2), \Delta Y)$$

between the functor $(-) \amalg (-) \colon \mathcal{C}^2 \longrightarrow \mathcal{C}$ and $\Delta \colon \mathcal{C} \longrightarrow \mathcal{C}^2$.

Exercise 1.19. What is the coproduct in $\mathcal{G}\mathrm{rp}$? Is this different from the coproduct in $\mathcal{A}\mathrm{b}\mathcal{G}\mathrm{rp}$?

Exercise 1.20. Let X be a set, and consider the functor $\mathcal{S}\mathrm{et}(X, -) \colon \mathcal{S}\mathrm{et} \longrightarrow \mathcal{S}\mathrm{et}$ that takes a set Y to the set $\mathcal{S}\mathrm{et}(X, Y)$ of all functions from X to Y. What is the left adjoint of $\mathcal{S}\mathrm{et}(X, -)$?