MATH 304: HOMEWORK 9 (due 4/26)

1. Consider the ring $R = \{a + bi\sqrt{5} \mid a, b \in \mathbb{Z}\}$.

   (i) Prove that the only units in R are the numbers 1 and -1.

   If $\alpha, \beta \in R$, we say that β divides α, written $\beta \mid \alpha$, if there exists $\gamma \in R$ such that $\alpha = \beta \cdot \gamma$. We say that $\alpha \in R$ is prime if the only divisors of α are ± 1 and $\pm\alpha$, i.e. units and unit multiples of α (just like for $\mathbb{Z}$ and $\mathbb{Z}[i]$).

   (ii) Prove that the number 2 is prime in R.

   (iii) The prime number 2 obviously divides the product

   $$6 = (1 + i\sqrt{5})(1 - i\sqrt{5}).$$

   Show that 2 does not divide either of the factors $1 + i\sqrt{5}$ or $1 - i\sqrt{5}$. Thus R does not satisfy the CRUCIAL LEMMA (if π is a prime and $\pi \mid \alpha \cdot \beta$, then $\pi \mid \alpha$ or $\pi \mid \beta$).

   (iv) Show that elements of R do NOT have unique factorization into primes by verifying that all of the numbers appearing in the two distinct factorizations

   $$6 = 2 \cdot 3 = (1 + i\sqrt{5})(1 - i\sqrt{5})$$

   are prime in R.

   (v) Find another number in R that has two distinct factorizations by finding distinct primes π_1, π_2, π_3 and π_4 such that $\pi_1 \cdot \pi_2 = \pi_3 \cdot \pi_4$.

2. Find four solutions in positive integers to the equation

   $$x^2 - 5y^2 = 1.$$

3. The number

   $$\gamma = \frac{1 + \sqrt{5}}{2} = 1.61803398874939\ldots$$

   is called the *golden ratio*.

   (a) For each $y \leq 20$, find the integer x making $|x - y\gamma|$ as small as possible. Which rational number x/y with $y \leq 20$ most closely approximates γ?

   (b) Find at least five instances of the golden ratio outside of mathematics proper: i.e. in nature, art, architecture, etc.

4. Consider the sequence of numbers $r_1, r_2, r_3, \ldots$ determined by the recursive definition

   $$r_1 = 1, \quad r_2 = 1 + \tfrac{1}{r_1}, \quad r_3 = 1 + \tfrac{1}{r_2}, \quad \ldots \quad r_n = 1 + \tfrac{1}{r_{n-1}}, \quad \ldots$$

   (a) Compute the values $r_1, \ldots, r_{10}$ (you should get $r_{10} = 89/55$).

(b) Let γ be the golden ratio. Compute the differences

$$|r_1 - \gamma|, \quad |r_2 - \gamma|, \quad \dots \quad |r_{10} - \gamma|$$

as decimals. Do you notice anything?

(c) Suppose that the sequence r_n converges to some number r, i.e. $r = \lim_{n\to\infty} r_n$. Use the recursive definition of r_n to argue that r satisfies the equation $r = 1 + \frac{1}{r}$.

(d) Suppose that $r > 0$ is a number satisfying the equation $r = 1 + \frac{1}{r}$. Prove that $r = \gamma$. We may then conclude from (c) that $\lim_{n\to\infty} r_n = \gamma$.

(e) Look again at the numerators and denominators of the fractions $r_1, \dots, r_{10}$. Do you recognize these numbers? If you can see a general pattern, try to prove it.

5. For each of the following equations, either find a solution (x, y) in positive integers, or explain why no such solution can exist.

   (i) $x^2 - 11y^2 = 7$

   (ii) $x^2 - 11y^2 = 433$

   (iii) $x^2 - 11y^2 = 3$

6. Find the units of the ring $R = \{a + b\sqrt{3} \mid a, b \in \mathbb{Z}\}$. (Hint: define a norm function on R by
$$N(a + b\sqrt{3}) = a^2 - 3b^2.$$
Show that the norm is multiplicative $N(\alpha\beta) = N(\alpha)N(\beta)$, then argue that if α is a unit in R, then $N(\alpha) = 1$.)