MATH 304: HOMEWORK 8 (due 4/19)

1. Suppose that $N = p_1^{e_1} \cdots p_k^{e_k}$ is a positive integer. Define the Jacobi symbol $\left(\frac{a}{N}\right)$ as the following product of Legendre symbols:

$$\left(\frac{a}{N}\right) = \left(\frac{a}{p_1}\right)^{e_1} \cdots \left(\frac{a}{p_k}\right)^{e_k}.$$

Let M and N be odd positive integers. Prove that:

$$\left(\frac{M}{N}\right)\left(\frac{N}{M}\right) = (-1)^{\frac{M-1}{2} \cdot \frac{N-1}{2}}.$$

2. Use Fermat's method of descent to write the prime $p = 12049$ as the sum of two squares, starting from the equation

$$557^2 + 55^2 = 26 \cdot 12049.$$

3. Suppose that $p > 5$ is prime and that p can be written in the form $p = a^2 + 5b^2$. Prove that

$$p \equiv 1 \text{ or } 9 \pmod{20}.$$

4. (i) Show that if n is a sum of three squares, then $n \not\equiv 7 \pmod 8$.

   (ii) Suppose n can be written as the sum of three squares $n = x^2 + y^2 + z^2$ and that 4 divides n. Prove that x, y and z are even.

   (iii) Show that if $n = 4^m(8k+7)$ with $m \geq 1$, then n is not the sum of three squares. (Gauss proved that all other positive integers n can be written as the sum of three squares.)

5. For each part, check whether the Gaussian integer α divides the Gaussian integer β and, if it does, find the quotient:

   (i) $\alpha = 3 + 5i$, $\beta = 11 - 8i$.

   (ii) $\alpha = 2 - 3i$, $\beta = 4 + 7i$.

   (iii) $\alpha = 3 - 5i$, $\beta = 3 - 39i$.

6. An element α of a ring R is called a unit if there exists an element $\beta \in R$ such that $\alpha \cdot \beta = 1$. In other words, α is a unit if it has a multiplicative inverse. It is common to write $R^\times$ for the set of units in a ring R. For each of the following rings, describe the set of units $R^\times$ (you may take for granted that $+$ and $\cdot$ give these sets the structure of a ring).

(i) $R_1 = \{a + bi\sqrt{2} \mid a, b \in \mathbb{Z}\}$

(ii) $R_2 = \{a + b\omega \mid a, b \in \mathbb{Z}\}$, where $\omega = -\frac{1}{2} + \frac{1}{2}i\sqrt{3}$.

(iii) $R_3 = \{\frac{a}{b} \mid a, b \in \mathbb{Z} \text{ and } p \nmid b\}$.

7. Factor each of the following Gaussian integers into a product of Gaussian primes: $91 + 63i$, 975 and $53 + 62i$.

8. Choose one of the following two statements and write a one-page essay defending it. Do not just reiterate the statement over and over using different terms. Give at least three specific reasons why your statement is true and the opposing statement is incorrect (or, if you have adventurous metaphysical opinions, attempt to synthesize the two opinions). You may want to employ empirical reasoning (deductions based on observations about the world or your personal experience) and/or *a priori* reasoning (arguments based on abstract first principles).

   Statement 1: Mathematics already exists and is merely discovered by people (in the same sense that Pluto existed before it was discovered in 1930).

   Statement 2: Mathematics is a creation invented by people to describe the world (and possibly having nothing to do with the real world). Mathematics does not exist independently of human thought.