MATH 304: HOMEWORK 7 (due 4/12)

1. Use the law of quadratic reciprocity to compute $\left(\frac{85}{101}\right)$, $\left(\frac{29}{541}\right)$, $\left(\frac{101}{1987}\right)$ and $\left(\frac{31706}{43789}\right)$.

2. Let p be a prime satisfying $p \equiv 3 \pmod 4$ and suppose that a is a quadratic residue modulo p.

   (a) Show that $x = a^{(p+1)/4}$ is a solution to the congruence

   $$x^2 \equiv a \pmod p.$$

   This gives an explicit way to find square roots modulo p when $p \equiv 3 \pmod 4$.

   (b) Find a solution x satisfying $1 \leq x \leq 787$ to the congruence

   $$x^2 \equiv 7 \pmod{787}.$$

3. Let p be a prime satisfying $p \equiv 5 \pmod 8$ and suppose that a is a quadratic residue modulo p.

   (a) Show that one of the values

   $$x = a^{(p+3)/8} \qquad \text{or} \qquad x = 2a \cdot (4a)^{(p-5)/8}$$

   is a solution to the congruence $x^2 \equiv a \pmod p$.

   (b) Find a solution x satisfying $1 \leq x \leq 541$ to the congruence $x^2 \equiv 5 \pmod{541}$.

   (c) Find a solution x satisfying $1 \leq x \leq 652$ to the congruence $x^2 \equiv 13 \pmod{653}$.

4. (a) Use quadratic reciprocity to find all of the primes for which 3 is a quadratic residue.
   (b) Do the same for 15.

5. Let (RR) denote the number of pairs $(n, n+1)$ in the set $1, 2, \ldots, p-1$ such that n and $n+1$ are both quadratic residues modulo p. let (RN) denote the number of pairs $(n, n+1)$ in the set $1, 2, \ldots, p-1$ such that n is a quadratic residue and $n+1$ is a quadratic nonresidue modulo p. Define (RN) and (NN) similarly. Determine the sums $(RR)+(RN), (NR)+(NN), (RR)+(NR)$ and $(RN)+(NN)$.

6. Let p be an odd prime. Let $f(a)$ be a function defined for a prime to p satisfying the following properties:

   (i) $f(a)$ only takes the values ± 1.

   (ii) If $a \equiv b \pmod p$, then $f(a) = f(b)$.

   (iii) $f(ab) = f(a)f(b)$ for all a and b.

Show that either $f(a) = 1$ for all a or that $f(a) = \left(\frac{a}{p}\right)$.

7. Let a be a nonzero integer, let p and q be odd primes that do not divide a and that satisfy

$$p \equiv q \pmod{4|a|}.$$

Show that $\left(\frac{a}{p}\right) = \left(\frac{a}{q}\right)$.

(Hint: write $a = \pm n^2 b$ where b is square-free (i.e. if $p \mid b$ then $p^2 \nmid b$). Then apply quadratic reciprocity to every prime divisor of b.)