MATH 304: HOMEWORK 4 (due 3/8)

1. Recall that the sigma function $\sigma(k)$ is defined to be the sum of all the divisors of k. Show that if m and n are relatively prime, then $\sigma(mn) = \sigma(m)\sigma(n)$.

2. Recall that n is a perfect number if n is equal to the sum of all proper divisors of n. Suppose that we looked at the product of divisors instead of the sum. Thus call n a *product perfect* number if n is equal to the product of all proper divisors of n. For example,
$$6 = 1 \cdot 2 \cdot 3,$$
so 6 is product perfect. (a) List all product perfect numbers between 2 and 50. (b) Find a nice criterion that exactly determines the product perfect numbers, and prove that your criterion works.

3. Show that if $a^n + 1$ is prime for $a \geq 2$ and $n \geq 1$, then n must be a power of 2. (Historical note: Fermat conjectured that every number of the form $2^{2^k} + 1$ is prime. We call prime number of this form Fermat primes.)

4. Let p be an odd prime that divides $a^{2^n} + 1$. Show that $p \equiv 1 \pmod{2^{n+1}}$. Hint: consider the order of $a \pmod p$ in the multiplicative group $(\mathbb{Z}/p\mathbb{Z})^\times$. (Historical note, continued: Euler used this method to find a divisor of $2^{32} + 1$, thus proving that it was not prime. This disproved Fermat's conjecture that every number of the form $2^{2^k} + 1$ was prime.)

5. Decode the following message, which was sent using the modulus $m = 7081$ and the exponent $k = 1789$ (note that you will first need to factor m).
$$5192 \quad 2604 \quad 4222$$

6. Suppose that $m = pq$, and $\varphi = (p-1)(q-1)$. FInd a formula for p and q, in terms of m and φ. Supposing that $m = 39,247,771$ is the product of two distinct primes, deduce the factors of m from the information that $\varphi(m) = 39,233,944$.