MATH 304: HOMEWORK 2 (due 2/21)

1. Find all of the distinct solutions modulo m (i.e. "incongruous" solutions) to the following congruences:
   (a) $8x \equiv 6 \pmod{14}$
   (b) $35x \equiv 14 \pmod{21}$
   (c) $x^2 \equiv 2 \pmod{7}$
2. Show that the following divisibility tests work:
   (a) The number a is divisible by 4 if and only if its last two digits are divisible by 4.
   (b) The number a is divisible by 3 if and only if the sum of its digits is divisible by 3.
   (c) The number a is divisible by 11 if and only if the alternating sum of its digits is divisible by 11.
3. Create your own test for divisibility by 8 in a style similar to the previous problem.
4. In this problem we will find the value of $(p-1)! \pmod{p}$.
   (a) Compute $(p-1)! \pmod{p}$ for small primes p, find a pattern and make a conjecture for the general case.
   (b) Prove your conjecture.
   (c) What can you say about $(n-1)! \pmod{n}$ when n is not prime?
5. Let p be an odd prime. In this problem, we will show that the equation
$$x^2 \equiv -1 \pmod{p}$$
   has solutions if and only if $p \equiv 1 \pmod{4}$ (the case $p = 2$ is trivial).
   (a) Suppose that $p \equiv 1 \pmod{4}$. Using your answer to 4(b), prove that
$$x = \left(\frac{p-1}{2}\right)!$$
   is a solution to the equation $x^2 \equiv -1 \pmod{p}$.
   (b) Conversely, suppose that x satisfies $x^2 \equiv -1 \pmod{p}$. Deduce that $p \equiv 1 \pmod{4}$.
6. A composite number m is called a *Carmichael number* if the congruence $a^{m-1} \equiv 1 \pmod{m}$ is true for every number a with $\mathrm{GCD}(a, m) = 1$.
   (a) Verify that $m = 561 = 3 \cdot 11 \cdot 17$ is a Carmichael number. [Hint: don't do this by brute force. Use a result from class.]
   (b) Try to find another Carmichael number.