MATH 304: HOMEWORK 10 (due 5/3)

1. Compute the first 10 terms in the continued fraction expressions for $\sqrt{3}$ and $\sqrt{5}$. In each case, find the period of the continued fraction and prove that it really does repeat with that periodicity.

2. Find the value of each of the following periodic continued fractions. Express your answer in the form $\frac{r+s\sqrt{D}}{t}$, where r, s, t, D are integers.

   (a) $[\overline{1,2,3}]$

   (b) $[3,\overline{2,1}]$

   (c) $[1,2,\overline{1,3,4}]$

3. Let α be an irrational number and let $\frac{p_n}{q_n}$ be the n^{th} convergent in the continued fraction expression for α. For each of the following values of α, make a table listing the values of the quantity

$$q_n|p_n - q_n\alpha| \qquad \text{for } n = 1, 2, \ldots, N.$$

   (a) $\alpha = \sqrt{2}$ up to $N = 8$.

   (b) $\alpha = \sqrt[3]{2}$ up to $N = 7$.

   (c) $\alpha = \pi$ up to $N = 5$.

   (d) Your data in (a) suggests that $|p_n - q_n\sqrt{2}|$ approaches a limit as $n \to \infty$. Find the limiting value and prove that it is indeed the limit.

   (e) Recall that Dirichlet's Diophantine approximation theorem says that there are infinitely many pairs of positive integers (x, y) such that

$$|x - y\alpha| < \frac{1}{y}.$$

   Your data above suggest that for every n, the pair (p_n, q_n) satisfies this inequality. Prove that this is true in general.

4. Consider the continued fraction

$$\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cdots}}}} = [1,1,1,1,\ldots] = [\overline{1}].$$

   (a) Compute the first 10 convergents of the continued fraction (Hint: you have done this before).

   (b) Argue that this continued fraction converges to the golden ratio $\gamma = \frac{1+\sqrt{5}}{2}$.

(c) Prove that the n^{th} convergent of the continued fraction is given by the ratio of successive terms in the Fibonacci sequence:

$$\frac{p_n}{q_n} = \frac{F_n}{F_{n-1}}.$$

(d) Conclude that the limit of the ratios of successive terms in the Fibonacci sequence is the golden ratio:

$$\lim_{n\to\infty} \frac{F_n}{F_{n-1}} = \gamma.$$

5. In class, we found the following formula for a periodic continued fraction of period 1:

$$[a, \overline{b}] = \frac{2a-b}{2} + \frac{\sqrt{b^2+4}}{2}$$

Find a similar formula for the number whose continued fraction expansion is $[a, \overline{b, c}]$. (Hint: to check your answer, notice that if you let $c = b$, you should recover the formula from class!)

6. (a) Used continued fractions to find a solution (x, y) to the generalized Pell's equation $x^2 - 29y^2 = -1$.

   (b) Use your answer to (a) to find a solution to Pell's equation $x^2 - 29y^2 = 1$.