MATH 304: HOMEWORK 1 (due 2/9)

1. Use the Euclidean algortihm to find the greatest common divisor of 1109 and 4999. Then find integers x and y that satisfy the equation

$$1109x + 4999y = \mathrm{GCD}(1109, 4999).$$

2. Read chapter 3 of Silverman (see the link on the course website for a pdf). (a) The equation

$$x^2 + y^2 = 2$$

describes a circle in the xy-plane. Use the lines through the the point $(1, 1)$ to describe all of the points on this circle whose coordinates are rational numbers. (b) What goes wrong if you try the same procedure for the circle $x^2 + y^2 = 3$?

3. Prove by induction that $4^{2n+1} + 3^{n+2}$ is divisible by 13.

4. The Fibonacci sequence $\{f_n\}$ is defined recursively by the equations:

$$\begin{aligned} f_1 &= 1 \\ f_2 &= 1 \\ f_n &= f_{n-1} + f_{n-2} \end{aligned}$$

Thus the first few terms of the sequence are:

$$1, 1, 2, 3, 5, 8, 13, 21, 34, \dots .$$

Prove that ever pair of consecutive entries f_n, f_{n+1} in the Fibonacci sequence are relatively prime.

5. (a) Show that if $m < n$, then $a^{2^m} + 1$ is a divisor of $a^{2^n} - 1$. (b) Further show that

$$\mathrm{GCD}(a^{2^m} + 1, a^{2^n} + 1) = \begin{cases} 1 & \text{if } a \text{ is even} \\ 2 & \text{if } a \text{ is odd.} \end{cases}$$

6. Give an example of three integers that are relatively prime but are not pairwise relatively prime.

7. Welcome to $\mathbb{M}$-world, where the only numbers are $\mathbb{M}$-numbers: the positive integers that leave a remainder of 1 when divided by 4. In other words, $\mathbb{M}$-world is:

$$\{1, 5, 9, 13, 17, 21, \dots\} = \{4t + 1 \mid t = 0, 1, 2, 3 \dots\}$$

Notice that we cannot add numbers in $\mathbb{M}$-world, as the sum of two numbers will not be an $\mathbb{M}$-number.

(a) Show that we can multiply numbers in $\mathbb{M}$-world, i.e. if x and y are two $\mathbb{M}$-numbers, then their product xy is an $\mathbb{M}$-number.

Let a and d be $\mathbb{M}$-numbers. We say that d $\mathbb{M}$-divides a if $a = d \cdot q$ for some $\mathbb{M}$-number q. We say that a is an $\mathbb{M}$-prime number if its only $\mathbb{M}$-divisors are 1 and itself (but as usual we don't consider 1 to be an $\mathbb{M}$-prime).

(b) Find the first six $\mathbb{M}$-primes.

(c) Find an $\mathbb{M}$-number n that has two *different* factorizations as a product of $\mathbb{M}$-primes.