Mathematics 202: Midterm Exam 2 Name: SOLUTIONS
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This exam consists of five questions, each worth eight points. For each question, give your response on the same page as the question is displayed, using the back side if necessary. You must show your reasoning to receive full credit.

Circle your section below:

1. Cong Ma T 1:30 Krieger 308
2. Cong Ma T 3:00 Shaffer 302
3. Arash Karami Th 4:30 Krieger 309
4. Nima Moini Th 1:30 Krieger 308
5. Jordan Paschke T 1:30 Krieger 300
6. Kalina Mincheva T 4:30 Krieger 302
7. Jordan Paschke Th 1:30 Hodson 211
8. Junyan Zhu Th 3 Ames 234
9. Junyan Zhu Th 4:30 Krieger 304

Problem #	Score
1	/ 8
2	/ 8
3	/ 8
4	/ 8
5	/ 8
TOTAL	/ 40

Do not turn this page until you are told to begin.

1. Compute the following integrals:

(a) $\iint_D 2x \, dxdy$ where $D \subset \mathbb{R}^2$ is the triangle with vertices $(1,3), (2,3)$, and $(2,0)$.

Writing the triangle as a y-simple region, it consists of points from $x = 1$ to $x = 2$

$$1 \le x \le 2$$

between the lines $y = -3x + 6$ and $y = 3$:

$$-3x + 6 \le y \le 3$$

Thus the double integral is

$$\begin{aligned}
\iint_D 2x \, dxdy &= \int_1^2 \int_{-3x+6}^3 2x \, dydx \\
&= \int_1^2 2x \cdot y \Big|_{y=-3x+6}^3 dx \\
&= \int_1^2 \big(2x \cdot 3 - 2x(-3x+6)\big) \, dx \\
&= \int_1^2 (6x^2 - 6x) \, dx \\
&= (2x^3 - 3x^2)\Big|_{x=1}^2 = 16 - 12 - 2 + 3 = 5
\end{aligned}$$

(b) $\displaystyle\int_0^1 \int_{\sqrt{y}}^1 e^{x^3}\, dxdy$

The key to this problem is to change the order of integration. The region of integrationas an x-simple region is given by the inequalities

$$\begin{aligned} 0 \le y &\le 1 \\ \sqrt{y} \le x &\le 1 \end{aligned}$$

This is the area in the plane from $x = 0$ to $x = 1$ between the line $y = x^2$ and the x-axis, i.e. as a y-simple region it is:

$$\begin{aligned} 0 \le x &\le 1 \\ 0 \le y &\le x^2 \end{aligned}$$

Therefore the double integral is

$$\begin{aligned} \int_0^1 \int_{\sqrt{y}}^1 e^{x^3}\, dxdy &= \int_0^1 \int_0^{x^2} e^{x^3}\, dydx \\ &= \int_0^1 (x^2 e^{x^3} - 0 \cdot e^{x^3})\, dx \\ &= \frac{1}{3} e^{x^3} \Big|_{x=0}^1 = \frac{1}{3}(e-1) \end{aligned}$$

2. Consider the vector field $F(x, y, z) = (2xy + yz, 2xy + xz, -2xz - 2yz + xy)$.

(a) Calculate the curl of F.

$$\begin{aligned}
\nabla \times F &= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \times F \\
&= \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (2xy + yz) & (2xy + xz) & (-2xz - 2yz + xy) \end{bmatrix} \\
&= \mathbf{i}(-2z + x - x) - \mathbf{j}(-2z + y - y) + \mathbf{k}(2y + z - 2x - z) \\
&= (-2z, 2z, 2y - 2x)
\end{aligned}$$

(b) Calculate the divergence of F.

$$\begin{aligned}
\nabla \cdot F &= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot F \\
&= \frac{\partial}{\partial x}(2xy + yz) + \frac{\partial}{\partial y}(2xy + xz) + \frac{\partial}{\partial z}(-2xz - 2yz + xy) \\
&= 2y + 2x + (-2x - 2y) = 0
\end{aligned}$$

(c) Is F conservative? In other words, is there a potential function $V : \mathbb{R}^3 \longrightarrow \mathbb{R}$ such that $\nabla V = F$?

No, F is not conservative. If $F = \nabla V$ for some V, then the curl of F would be

$$\nabla \times (\nabla V) = 0$$

Since the curl is nonzero, F cannot be conservative.

3. Find the maximum and minimum value of the function

$$f(x, y, z) = xy + z$$

on the solid ball of radius 2:

$$x^2 + y^2 + z^2 \leq 4.$$

The gradient of f is:

$$\nabla f = (y, x, 1)$$

Since this is always nonzero, f has no global critical points. In particular, there are no critical points on the inside of the ball. Thus we only need to check the critical point on the boundary of the ball, i.e. subject to the constraint

$$g(x, y, z) = x^2 + y^2 + z^2 = 4.$$

Lagrange's multiplier method gives the equation

$$(y, x, 1) = \nabla f = \lambda \nabla g = (2x, 2y, 2z)$$

Along with the constraint equation, we have the system of equations

$$\begin{cases} y = 2\lambda x \\ x = 2\lambda y \\ 1 = 2\lambda z \\ x^2 + y^2 + z^2 = 4 \end{cases}$$

Plugging the first equation into the second, we get

$$x = 4\lambda^2 x$$

This equation has three solutions: (I) $x = 0$, (II) $\lambda = \frac{1}{2}$, or (III) $\lambda = -\frac{1}{2}$.

In case (I), we find from the first equation that $y = 0$, so by the fourth equation, $z = \pm 2$. This gives the critical points $(0, 0, \pm 2)$. The value of f on these critical points is $f(0, 0, 2) = 2$ and $f(0, 0, -2) = -2$.

In case (II), we find from the third equation that $z = 1$, from the first equation that $y = x$, and from the fourth equation that $2x^2 + 1 = 4$. Therefore $x = \pm\sqrt{3/2}$. This gives the critical points $(\pm\sqrt{3/2}, \pm\sqrt{3/2}, 1)$. The value of f on these critical points is $f(\pm\sqrt{3/2}, \pm\sqrt{3/2}, 1)) = 5/2$.

In case (III), we find from the third equation that $z = -1$, from the first equation that $y = -x$, and from the fourth equation that $2x^2 + 1 = 4$. Therefore $x = \pm\sqrt{3/2}$ and $y = \mp\sqrt{3/2}$. This gives the critical points $(\pm\sqrt{3/2}. \mp \sqrt{3/2}, -1)$. The value of f on these critical points is $f(\pm\sqrt{3/2}. \mp \sqrt{3/2}, -1) = -5/2$.

By comparing the values of f on the critical points, we find that the maximum value of f is $5/2$ at $(\sqrt{3/2}, \sqrt{3/2}, 1)$ and $(-\sqrt{3/2}, -\sqrt{3/2}, 1)$, and that the minimum value of f is $-5/2$ at $(\sqrt{3/2}, -\sqrt{3/2}, -1)$ and $(-\sqrt{3/2}, \sqrt{3/2}, -1)$.

4. Let $W \subset \mathbb{R}^3$ be the solid region defined by the inequalities

$$W\colon \begin{cases} 1 \leq x^2 + y^2 + z^2 \leq 9 \\ y \geq 0 \\ z \geq 0 \end{cases}$$

Evaluate the following triple integral:

$$\iiint_W e^{-(x^2+y^2+z^2)^{3/2}}\, dxdydz.$$

The region W lies between the spheres of radius 1 and 3, above the xy-plane and in front of the xz-plane. If you picture W, it is a quarter slice of the spherical region from radius 1 to radius 3. In spherical coordinates, this is given by the inequalities

$$1 \leq \rho \leq 3$$
$$0 \leq \theta \leq \pi$$
$$0 \leq \phi \leq \frac{\pi}{2}$$

Using the change to spherical coordinates function $T(\rho, \theta, \phi) = (\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi)$, we note that

$$x^2 + y^2 + z^2 = \rho^2,$$

make sure to remember the warp factor $\rho^2 \sin\phi$, and then compute:

$$\begin{aligned}
\iiint_W e^{-(x^2+y^2+z^2)^{3/2}}\, dxdydz &= \int_0^{\pi/2}\int_0^{\pi}\int_1^3 e^{-(\rho^2)^{3/2}} \rho^2 \sin\phi\, d\rho d\theta d\phi \\
&= \int_0^{\pi/2}\int_0^{\pi}\int_1^3 e^{-\rho^3} \rho^2 \sin\phi\, d\rho d\theta d\phi \\
&= \int_0^{\pi/2}\int_0^{\pi} \left(-\frac{1}{3}e^{-\rho^3}\right)\Big|_{\rho=1}^{3} \sin\phi\, d\theta d\phi \\
&= \int_0^{\pi/2} \pi\left(-\frac{1}{3}e^{-27} + \frac{1}{3}e^{-1}\right) \sin\phi\, d\phi \\
&= \frac{\pi}{3}(-e^{-27} + e^{-1})(-\cos\phi)\Big|_{\phi=0}^{\pi/2} \\
&= \frac{\pi}{3}(-e^{-27} + e^{-1})(0 - (-1)) \\
&= \frac{\pi}{3}(-e^{-27} + e^{-1})
\end{aligned}$$

5. Let D be the region in the xy-plane consisting of those points inside the ellipse

$$x^2 + \frac{y^2}{4} = 1$$

that lie above the x-axis, i.e. for which $y \geq 0$.

Evaluate the double integral $\iint_D (x+y)\, dxdy$.

The region D is the top half of the ellipse with major axis along the y-axis from $y = -2$ to $y = 2$ and minor axis along the x-axis from $x = -1$ to $x = 1$. As I explained in lecture, such an ellipse is parametrized by a stretched version of the polar coordinates function, in this case:

$$T(r, \theta) = (r\cos\theta, 2r\sin\theta)$$

Integrating over the top half of the ellipse corresponds to the region

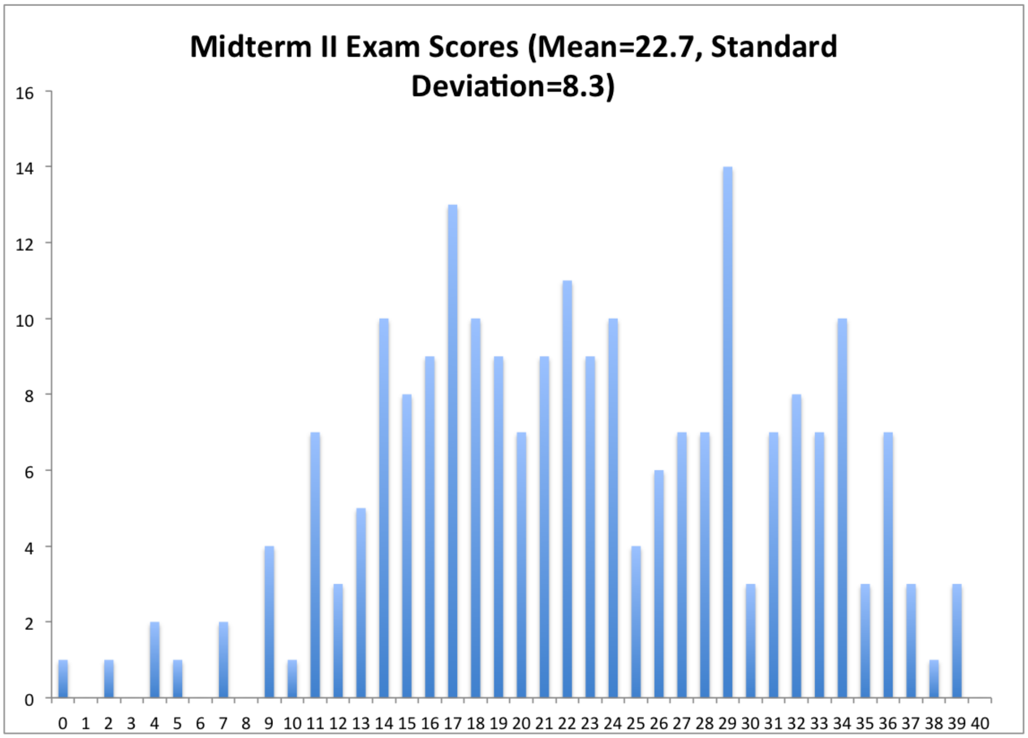
$$0 \leq r \leq 1$$
$$0 \leq \theta \leq \pi$$

The warp factor for T is

$$\begin{aligned} \frac{\partial(x,y)}{\partial(r,\theta)} = \det DT_{(r,\theta)} &= \det \begin{bmatrix} \cos\theta & -r\sin\theta \\ 2\sin\theta & 2r\cos\theta \end{bmatrix} \\ &= 2r(\cos^2\theta + \sin^2\theta) \\ &= 2r \end{aligned}$$

Applying the change of coordinates formula, we get:

$$\begin{aligned} \iint_D (x+y)\, dxdy &= \int_0^\pi \int_0^1 (r\cos\theta + 2r\sin\theta)2r\, drd\theta \\ &= \int_0^\pi \int_0^1 (\cos\theta + 2\sin\theta)2r^2\, drd\theta \\ &= \int_0^\pi (\cos\theta + 2\sin\theta)\frac{2}{3}r^3\Big|_{r=0}^{1}\, d\theta \\ &= \frac{2}{3}\int_0^\pi (\cos\theta + 2\sin\theta)\, d\theta \\ &= \frac{2}{3}(\sin\theta - 2\cos\theta)\Big|_{\theta=0}^{\pi} \\ &= \frac{2}{3}(0 - 2\cdot(-1)) - (0 - 2\cdot(1)) \\ &= \frac{8}{3} \end{aligned}$$


Midterm II Exam Scores (Mean=22.7, Standard Deviation=8.3)
0 2 4 6 8 10 12 14 16
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31 32 33 34 35 36 37 38 39 40