Math 202 – Calculus III   Midterm 2   04/09/2010

▶ **Your PRINTED name is:** Solutions

▶ **Please circle your section:** (one point off if you don't)

(1)	T 1:30	Hodson 216	McGonagle, Matthew
(2)	T 3:00	Bloomberg 168	McGonagle, Matthew
(3)	Th 4:30	Krieger 308	Lin, Longzhi
(4)	Th 1:30	Shaffer 300	Lin, Longzhi
(5)	Th 4:30	Krieger 300	Banerjee, Romie
(6)	Th 1:30	Dunning 205	Banerjee, Romie
(7)	Th 3:00	Bloomberg 168	Lin, Longzhi
(8)	Th 4:30	Krieger 302	Cutrone, Joseph

Grading

1	30
2	20
3	20
4	30
5	10

Total: ∞

▶ **Write out and SIGN the pledge:**

I pledge my honor that I have not violated the Honor Code during this examination.

Signature: Date:

▶ **This is a 50 minutes in-class closed book exam. This examination booklet contains 5 problems, including one bonus problem, on 7 sheets of paper including the front cover. Please detach the last page before exam, which is intended for use as scrap paper.**

1 (30 pts.) Consider the vector field $\vec{F} = \langle x, 2y, 3z \rangle$.

(1) Compute div$\vec{F}$.

Solution

$$\text{div}\vec{F} = \frac{\partial(x)}{\partial x} + \frac{\partial(2y)}{\partial y} + \frac{\partial(3z)}{\partial z} = 1 + 2 + 3 = 6.$$

(2) Compute curl$\vec{F}$.

Solution

$$\text{curl}\vec{F} = \langle \frac{\partial(3z)}{\partial y} - \frac{\partial(2y)}{\partial z}, \frac{\partial(x)}{\partial z} - \frac{\partial(3z)}{\partial x}, \frac{\partial(2y)}{\partial x} - \frac{\partial(x)}{\partial y} \rangle = \langle 0, 0, 0 \rangle.$$

(3) Is $\vec{F}$ the curl of another vector field? Explain why.

Solution No, because div$\vec{F} = 6 \neq 0$.

(4) Is $\vec{F}$ a gradient vector field? If yes, find out its potential functions; if no, explain why.

Solution Yes, since curl$\vec{F} = \vec{0}$. We find its potential f as follows:

First since $\frac{\partial f}{\partial x} = x$, we must have

$$f(x, y, z) = \int x dx = \frac{x^2}{2} + g(y, z)$$

for some function $g(y, z)$.

Second we take y, z derivatives of the previous equality, then it follows

$$2y = \frac{\partial f}{\partial y} = \frac{\partial(\frac{x^2}{2} + g(y,z))}{\partial y} = \frac{\partial g}{\partial y}, \qquad 3z = \frac{\partial f}{\partial z} = \frac{\partial(\frac{x^2}{2} + g(y,z))}{\partial z} = \frac{\partial g}{\partial z}.$$

It follows that $g(y, z) = \int 2y dy = y^2 + h(z)$ and $h'(z) = \frac{\partial g}{\partial z} = 3z$. So $h(z) = \frac{3z^2}{2} + C$. We conclude

$$f(x, y, z) = \int x dx = \frac{x^2}{2} + y^2 + \frac{3z^2}{2} + C.$$

2 (20 pts.) Let D be the triangle in $\mathbb{R}^2$ with vertices $(0,0),(1,0)$ and $(1,1)$. Evaluate the double integral $\iint_D (x+y)dxdy$, by the following two methods:

(1) Rewrite the double integral as an iterated integral over a y-simple region and evaluate it.

Solution As a y-simple region, D is given by $0 \le x \le 1, 0 \le y \le x$. So

$$\begin{aligned}\iint_D (x+y)dxdy &= \int_0^1 \int_0^x (x+y)dydx \\ &= \int_0^1 (xy + \frac{y^2}{2})|_{y=0}^{y=x} dx \\ &= \int_0^1 \frac{3x^2}{2} dx \\ &= \frac{x^3}{2}|_{x=0}^{x=1} = \frac{1}{2}.\end{aligned}$$

(2) Rewrite the double integral as an iterated integral over a x-simple region and evaluate it.

Solution As a x-simple region, D is given by $0 \le y \le 1, y \le x \le 1$. So

$$\begin{aligned}\iint_D (x+y)dxdy &= \int_0^1 \int_y^1 (x+y)dxdy \\ &= \int_0^1 (\frac{x^2}{2} + yx)|_{x=y}^{x=1} dy \\ &= \int_0^1 (\frac{1}{2} + y - \frac{3y^2}{2})dy \\ &= (\frac{1}{2}y + \frac{1}{2}y^2 - \frac{1}{2}y^3)|_{y=0}^{y=1} = \frac{1}{2}.\end{aligned}$$

3 (20 pts.) Still let D be the triangle in $\mathbb{R}^2$ with vertices $(0,0), (1,0)$ and $(1,1)$. Evaluate the integral $\iint_D \frac{x}{x^2+y^2}dxdy$ by converting to polar coordinates.

Solution The boundary of the triangle D are the lines $y=0, y=x$ and $x=1$. If we use the polar coordinates, these equations become $r\sin\theta = 0, r\sin\theta = r\cos\theta$ and $r\cos\theta = 1$, i.e. $\theta = 0, \theta = \frac{\pi}{4}$ and $r = \frac{1}{\cos\theta}$.

It follows that in polar coordinates, D is given by

$$0 \leq \theta \leq \frac{\pi}{4}, \qquad 0 \leq r \leq \frac{1}{\cos\theta}.$$

So

$$\begin{aligned}
\iint_D \frac{x}{x^2+y^2}dxdy &= \int_0^{\frac{\pi}{4}} \int_0^{\frac{1}{\cos\theta}} \frac{r\cos\theta}{r^2} rdrd\theta \\
&= \int_0^{\frac{\pi}{4}} \int_0^{\frac{1}{\cos\theta}} \cos\theta drd\theta \\
&= \int_0^{\frac{\pi}{4}} d\theta \\
&= \frac{\pi}{4}
\end{aligned}$$

4 (30 pts.) (1) Find the volume of the region V which is bounded by the surface $z = 4 - x^2 - y^2$ and the xy plane.

Solution V has a rotating symmetry, and can be described in cylindrical system as

$$0 \le r \le 2, \qquad 0 \le \theta \le 2\pi, \qquad 0 \le z \le 4 - r^2.$$

So

$$\begin{aligned}
\text{Volume} &= \iiint_V 1 dxdydz \\
&= \int_0^2 \int_0^{2\pi} \int_0^{4-r^2} r dz d\theta dr \\
&= \int_0^2 \int_0^{2\pi} (4r - r^3) d\theta dr \\
&= 2\pi \int_0^2 (4r - r^3) dr \\
&= 2\pi (2r^2 - \frac{r^4}{4})|_{r=0}^{r=2} \\
&= 8\pi.
\end{aligned}$$

(2) Compute the average of $f(x, y) = 4 - x^2 - y^2$ where the surface $z = f(x, y)$ is above the xy plane.

Solution The region where $z = f(x, y)$ is above the xy plane is $D = \{(x, y) \mid x^2 + y^2 \le 4\}$, the disc with radius 2 and center $(0, 0)$. By definition, the average of $f = f(x, y)$ over D is

$$[f]_{av} = \frac{\iint_D f(x, y) dxdy}{\text{Area}(D)}.$$

Note that the integral in the numerator is exactly the volume we computed in the first part, which equals 8π, and the area in the denominator is 4π. We conclude

$$[f]_{av} = \frac{8\pi}{4\pi} = 2.$$

5 (10 pts.) **(This is only a bonus problem. Do other problems first!)**

A helical thin wire $\mathcal{C}$ is parametrized by

$$\vec{r}(t) = \langle \cos t, \sin t, t \rangle, 1 \leq t \leq 4.$$

Suppose the mass density of this wire at any point equals the square of the distance from that point to the origin: $\rho(x, y, z) = x^2 + y^2 + z^2$. Find the mass of this wire. (I know we have not define such integrals on curves yet. But, can you define it?)

$\boxed{\text{Solution}}$ Recall that the length of a curve is given by

$$\begin{aligned}\text{Length} &= \sum_i (\text{length of small pieces } \Delta s_i) \\ &= \lim_{\Delta t_i \to 0} \sum_i |\vec{r}\,'(t_i)| \Delta t_i = \int_a^b |\vec{r}\,'(t)| dt.\end{aligned}$$

Use the same idea, one can compute the mass via

$$\begin{aligned}\text{Mass} &= \sum_i (\text{mass of small pieces } \Delta m_i) \\ &= \lim_{\Delta s_i \to 0} \sum_i \rho(p_i^*) \Delta s_i \qquad \text{(since mass=density} \times \text{length)} \\ &= \lim_{\Delta t_i \to 0} \sum_i \rho(\vec{r}(t_i)) |\vec{r}\,'(t_i)| \Delta t_i = \int_a^b \rho(\vec{r}(t)) |\vec{r}\,'(t)| dt.\end{aligned}$$

It follows that for this problem,

$$\begin{aligned}\text{Mass} &= \int_1^4 ((\cos t)^2 + (\sin t)^2 + t^2) \sqrt{(-\sin t)^2 + (\cos t)^2 + 1^2} dt \\ &= \int_1^4 (1 + t^2) \sqrt{2} dt \\ &= \sqrt{2}\, (t + \frac{t^3}{3}) \Big|_{t=1}^{t=4} \\ &= 24\sqrt{2}.\end{aligned}$$

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