Math 202 – Calculus III

Practice Midterm 2

▶ **Your PRINTED name is:** **Solutions**

▶ **Please circle your section:**

(1)	T 1:30	Hodson 216	McGonagle, Matthew
(2)	T 3:00	Bloomberg 168	McGonagle, Matthew
(3)	Th 4:30	Krieger 308	Lin, Longzhi
(4)	Th 1:30	Shaffer 300	Lin, Longzhi
(5)	Th 4:30	Krieger 300	Banerjee, Romie
(6)	Th 1:30	Dunning 205	Banerjee, Romie
(7)	Th 3:00	Bloomberg 168	Lin, Longzhi
(8)	Th 4:30	Krieger 302	Cutrone, Joseph

Grading

1	**0**
2	**0**
3	**0**
4	**0**
5	**0**

Total: ∞

▶ **Write out and SIGN the pledge:**

I pledge my honor that I have not violated
the Honor Code during this examination.

Signature: Date:

▶ **This is a 50 minutes in-class closed book exam. This examination booklet contains 5 problems, including one bonus problem, on 7 sheets of paper including the front cover. Please detach the last page before exam, which is intended for use as scrap paper.**

1 (30 pts.) Suppose $\vec{F} = \langle ax^2y + y^3 + 1, 2x^3 + bxy^2 + 2\rangle$ is a gradient vector field.

(1) Find the values of a and b.

Solution Since $\vec{F}$ is a gradient vector field, we should have curl$\vec{F} = \vec{0}$.

By definition of the scalar curl,

$$\text{curl}\vec{F} = \frac{\partial(2x^3 + bxy^2 + 2)}{\partial x} - \frac{\partial(ax^2y + y^3 + 1)}{\partial y} = 6x^2 + by^2 - ax^2 - 3y^2.$$

So we must have

$$a = 6, \qquad b = 3.$$

(2) Find the potential functions of $\vec{F}$.

Solution We integrate to get

$$f(x, y) = \int (6x^2y + y^3 + 1)dx = 2x^3y + xy^3 + x + g(y).$$

Taking y derivative, we get

$$2x^3 + 3xy^2 + 2 = \frac{\partial f}{\partial y} = 2x^3 + 3xy^2 + g'(y).$$

It follows that $g'(y) = 2$, i.e. $g(y) = 2y + C$. So

$$f(x, y) = 2x^3y + xy^3 + x + 2y + C.$$

(3) Is $\vec{F}$ also the curl of a vector field? Explain why.

Solution $\vec{F}$ is not the curl of a vector field, since

$$\begin{aligned}\text{div}\vec{F} &= \frac{\partial(6x^2y + y^3 + 1)}{\partial x} + \frac{\partial(2x^3 + 3xy^2 + 2)}{\partial y} \\ &= 12xy + 6xy = 18xy \neq 0.\end{aligned}$$

2 (20 pts.) Change the order of the integration to evaluate the iterated integral

$$\int_0^1 \int_{\sqrt{x}}^1 \frac{1}{\sqrt{1+y^3}} dydx.$$

$\boxed{\text{Solution}}$ The domain of integration is the y-simple region $0 \le x \le 1, \sqrt{x} \le y \le 1$, which is also x-simple described by $0 \le y \le 1, 0 \le x \le y^2$. So

$$\begin{aligned}
\int_0^1 \int_{\sqrt{x}}^1 \frac{1}{\sqrt{1+y^3}} dydx &= \int_0^1 \int_0^{y^2} \frac{1}{\sqrt{1+y^3}} dxdy \\
&= \int_0^1 \frac{y^2}{\sqrt{1+y^3}} dy \\
&= \frac{1}{3} \int_0^1 \frac{1}{\sqrt{1+u}} du \qquad (\text{ we used } u = y^3) \\
&= \frac{2}{3} \sqrt{1+u}\Big|_{u=0}^{u=1} \\
&= \frac{2}{3}(\sqrt{2}-1).
\end{aligned}$$

3 (30 pts.) (1) Find the volume of the region inside the sphere $x^2 + y^2 + z^2 = 4$ and outside the cylinder $x^2 + y^2 = 1$.

Solution The region V above has a rotating symmetry, and is easily described via cylindrical coordinates:

$$-\sqrt{3} \le z \le \sqrt{3}, \quad 0 \le \theta \le 2\pi, \quad 1 \le r \le \sqrt{4 - z^2}.$$

So the volume is

$$\begin{aligned}
\text{Volume} &= \iiint_V 1dxdydz \\
&= \iiint_V rdrd\theta dz \\
&= \int_{-\sqrt{3}}^{\sqrt{3}} \int_0^{2\pi} \int_1^{\sqrt{4-z^2}} rdrd\theta dz \\
&= \pi \int_{-\sqrt{3}}^{\sqrt{3}} (3 - z^2)dz \\
&= 4\sqrt{3}\pi.
\end{aligned}$$

(2) Find the center of mass of this region assuming $\rho(x, y, z) = 1$.

Solution Two observations:

- The region V has a rotating symmetry, and the function ρ is also invariant under rotation with respect to z-axis. So the center of mass should lies on the z-axis.
- The region V is symmetric under the reflection with respect to the xy-plane, and the function ρ is also invariant under this reflection. So the center of mass should lies on the xy-plane.

It follows from these two observations that the center of mass must be the origin, $(0, 0, 0)$.

4 (20 pts.) Compute the area of the region in the first quadrant of the plane bounded by the curves $xy = 1, xy = 4, xy^3 = 3$ and $xy^3 = 9$.

Solution Obviously we should introduce the change of variables

$$u = xy, \quad v = xy^3$$

so the the region D in the problem is given by $1 \le u \le 4, 3 \le v \le 9$. The Jacobian determinant for this transform is

$$|J| = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \left|\frac{\partial(u,v)}{\partial(x,y)}\right|^{-1} = \left|\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right|^{-1} = |3xy^3 - xy^3|^{-1} = \frac{1}{2v}.$$

It follows

$$\begin{aligned}\text{Area} &= \iint_D 1dxdy \\ &= \int_1^4 \int_3^9 \frac{1}{2v} dvdu \\ &= \frac{3}{2}\int_3^9 \frac{1}{v}dv \\ &= \frac{3}{2} \ln v|_{v=3}^{v=9} \\ &= \frac{3}{2}(\ln 9 - \ln 3) \\ &= \frac{3}{2}\ln 3.\end{aligned}$$

5 (10 pts.) **(This is only a bonus problem. Do other problems first!)**

Suppose

$$T : D \to T(D), \quad (x, y) \mapsto (u(x, y), v(x, y))$$

and

$$T' : T(D) \to T'(T(D)), \quad (u, v) \mapsto (\alpha(u, v), \beta(u, v))$$

be two invertible C^1 transformations. Prove

$$\frac{\partial(\alpha, \beta)}{\partial(x, y)} = \frac{\partial(\alpha, \beta)}{\partial(u, v)} \cdot \frac{\partial(u, v)}{\partial(x, y)}.$$

(We used a three dimensional version of this in class, and I was sad that no one asked me why!)

Solution Recall that from the definition,

$$\frac{\partial(\alpha, \beta)}{\partial(u, v)} = \frac{\partial \alpha}{\partial u}\frac{\partial \beta}{\partial v} - \frac{\partial \alpha}{\partial v}\frac{\partial \beta}{\partial u}$$

and

$$\frac{\partial(u, v)}{\partial(x, y)} = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}.$$

So their product

$$\begin{aligned}\frac{\partial(\alpha, \beta)}{\partial(u, v)} \cdot \frac{\partial(u, v)}{\partial(x, y)} &= \left(\frac{\partial \alpha}{\partial u}\frac{\partial \beta}{\partial v} - \frac{\partial \alpha}{\partial v}\frac{\partial \beta}{\partial u}\right)\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right) \\ &= \frac{\partial \alpha}{\partial u}\frac{\partial \beta}{\partial v}\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial \alpha}{\partial v}\frac{\partial \beta}{\partial u}\frac{\partial u}{\partial y}\frac{\partial v}{\partial x} - \frac{\partial \alpha}{\partial u}\frac{\partial \beta}{\partial v}\frac{\partial u}{\partial y}\frac{\partial v}{\partial x} - \frac{\partial \alpha}{\partial v}\frac{\partial \beta}{\partial u}\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}.\end{aligned}$$

On the other hand,

$$\begin{aligned}\frac{\partial(\alpha, \beta)}{\partial(x, y)} &= \frac{\partial \alpha}{\partial x}\frac{\partial \beta}{\partial y} - \frac{\partial \alpha}{\partial y}\frac{\partial \beta}{\partial x} \\ &= \left(\frac{\partial \alpha}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial \alpha}{\partial v}\frac{\partial v}{\partial x}\right)\left(\frac{\partial \beta}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial \beta}{\partial v}\frac{\partial v}{\partial y}\right) \\ &\quad - \left(\frac{\partial \alpha}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial \alpha}{\partial v}\frac{\partial v}{\partial y}\right)\left(\frac{\partial \beta}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial \beta}{\partial v}\frac{\partial v}{\partial x}\right) \\ &= \frac{\partial \alpha}{\partial u}\frac{\partial \beta}{\partial v}\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial \alpha}{\partial v}\frac{\partial \beta}{\partial u}\frac{\partial u}{\partial y}\frac{\partial v}{\partial x} - \frac{\partial \alpha}{\partial u}\frac{\partial \beta}{\partial v}\frac{\partial u}{\partial y}\frac{\partial v}{\partial x} - \frac{\partial \alpha}{\partial v}\frac{\partial \beta}{\partial u}\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}.\end{aligned}$$

So we are not canceling something like $\partial(u, v)$ since $\frac{\partial(\alpha,\beta)}{\partial(u,v)}$ is not a quotient of two numbers. Instead, the identity we want to prove relies on the chain rule!

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