Mathematics 202: Midterm Exam 1 Name: SOLUTIONS
John Lind

This exam consists of five questions, each worth eight points. For each question, give your response on the same page as the question is displayed, using the back side if necessary. You must show your reasoning to receive full credit.

Circle your section below:

1. Cong Ma T 1:30 Krieger 308
2. Cong Ma T 3:00 Shaffer 302
3. Arash Karami Th 4:30 Krieger 309
4. Nima Moini Th 1:30 Krieger 308
5. Jordan Paschke T 1:30 Krieger 300
6. Kalina Mincheva T 4:30 Krieger 302
7. Jordan Paschke Th 1:30 Hodson 211
8. Junyan Zhu Th 3 Ames 234
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Problem #	Score
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TOTAL	/ 40

Do not turn this page until you are told to begin.

1. Consider the curve in $\mathbb{R}^3$ defined by the function:

$$\mathbf{c}(t) = (20t^2 - 117t)\mathbf{i} - (30\sqrt{t})\mathbf{j} + (2t^2 - 8t)\mathbf{k}.$$

(a) Find the velocity vector $\mathbf{v}$ of the curve at $t = 3$.

The velocity of the curve at time t is given by the derivative of the function $\mathbf{c}$ with respect to the time variable t:

$$\mathbf{c}'(t) = (40t - 117)\mathbf{i} - \frac{30}{2\sqrt{t}}\mathbf{j} + (4t - 8)\mathbf{k}.$$

At time $t = 3$, the velocity vector is:

$$\begin{aligned} \mathbf{v} = \mathbf{c}'(3) &= (40 \cdot 3 - 117)\mathbf{i} - \frac{30}{2\sqrt{3}}\mathbf{j} + (4 \cdot 3 - 8)\mathbf{k} \\ &= 3\mathbf{i} - (5\sqrt{3})\mathbf{j} + 4\mathbf{k} \\ &= (3, -5\sqrt{3}, 4) \qquad \text{[4 points]} \end{aligned}$$

(b) What angle does the velocity vector $\mathbf{v}$ from part (a) make with the y-axis? [4 points]

The angle θ between $\mathbf{v}$ and the y-axis is the angle between the vectors $\mathbf{v}$ and $\mathbf{j}$. We will find θ using the formula for the inner product in terms of length and angle:

$$\langle \mathbf{v}, \mathbf{j} \rangle = \|\mathbf{v}\| \|\mathbf{j}\| \cos\theta.$$

Plugging in the value $\mathbf{v} = (3, -5\sqrt{3}, 4)$ from part (a), we calculate that:

$$\langle \mathbf{v}, \mathbf{j} \rangle = \langle (3, -5\sqrt{3}, 4), (0, 1, 0) \rangle = -5\sqrt{3}.$$

Since $\mathbf{j} = (0, 1, 0)$ has length $\|\mathbf{j}\| = 1$,

$$\|\mathbf{v}\| \|\mathbf{j}\| = \|(3, -5\sqrt{3}, 4)\| = \sqrt{3^2 + (-5\sqrt{3})^2 + 4^2} = \sqrt{9 + 75 + 16} = \sqrt{100} = 10$$

Therefore,

$$\cos\theta = \frac{\langle \mathbf{v}, \mathbf{j} \rangle}{\|\mathbf{v}\| \|\mathbf{j}\|} = \frac{-5\sqrt{3}}{10} = -\frac{\sqrt{3}}{2},$$

so the angle between $\mathbf{v}$ and the y-axis is:

$$\theta = \frac{5\pi}{6} = 150^\circ \qquad \text{[4 points]}$$

2. Consider the function $f\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ defined by

$$f(x,y) = (x^3 e^{xy}, x^2y^2 - y)$$

(a) Find the Jacobian matrix $Df_{(1,1)}$ (i.e. the derivative) of f at the point $(1,1)$.
The matrix of partial derivatives is:

$$Df_{(x,y)} = \begin{bmatrix} 3x^2e^{xy} + x^3ye^{xy} & x^4e^{xy} \\ 2xy^2 & 2x^2y - 1 \end{bmatrix}$$

Evaluating at $(x,y) = (1,1)$, we find that the Jacobian matrix at $(1,1)$ is:

$$Df_{(1,1)} = \begin{bmatrix} 4e & e \\ 2 & 1 \end{bmatrix} \qquad \text{[3 points]}$$

Recall that the best affine approximation $T_{(1,1)}f$ of f at $(1,1)$ is defined in terms of the Jacobian as:

$$T_{(1,1)}f(x,y) = f(1,1) + L_{Df_{(1,1)}}(x-1, y-1),$$

where $L_{Df_{(1,1)}}$ is the linear function associated to the matrix $Df_{(1,1)}$.
(b) To evaluate $T_{(1,1)}f(2,1)$, we first compute that the linear function $L_{Df_{(1,1)}}$ is:

$$L_{Df_{(1,1)}}(x,y) = \begin{bmatrix} 4e & e \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4ex + ey \\ 2x + y \end{bmatrix} = (4ex + ey, 2x + y).$$

Therefore:

$$\begin{aligned} T_{(1,1)}f(2,1) &= f(1,1) + L_{Df_{(1,1)}}(2-1, 1-1) \\ &= (e,0) + L_{Df_{(1,1)}}(1,0) = (e,0) + (4e,2) = (5e,2). \qquad \text{[2 points]} \end{aligned}$$

(c) Let $g\colon \mathbb{R}^2 \to \mathbb{R}^3$ be defined by:

$$g(u, v) = (u, ve^u, -uv).$$

With f defined as in (a) and (b), compute the Jacobian matrix $D(g \circ f)_{(1,1)}$ using the chain rule.

The chain rule states that:

$$D(g \circ f)_{(1,1)} = Dg_{f(1,1)} \cdot Df_{(1,1)}$$

To evaluate the right hand side, we first need to compute the matrix of partial derivatives of $g(u, v)$:

$$Dg_{(u,v)} = \begin{bmatrix} 1 & 0 \\ ve^u & e^u \\ -v & -u \end{bmatrix}$$

Evaluating at $f(1, 1) = (e, 0)$, we get:

$$Dg_{f(1,1)} = Dg_{(e,0)} = \begin{bmatrix} 1 & 0 \\ 0 & e^e \\ 0 & -e \end{bmatrix}$$

Applying the chain rule, we find that:

$$D(g \circ f)_{(1,1)} = Dg_{f(1,1)} \cdot Df_{(1,1)} = \begin{bmatrix} 1 & 0 \\ 0 & e^e \\ 0 & -e \end{bmatrix} \begin{bmatrix} 4e & e \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 4e & e \\ 2e^e & e^e \\ -2e & -e \end{bmatrix} \qquad \text{[3 points]}$$

3. Consider the lines ℓ_1 and ℓ_2 parametrized by the equations

$$\ell_1(t) = (1,0,1) + t(1,2,-1)$$
$$\ell_2(s) = (6,-2,2) + s(-1,2,-1)$$

[In the 11am hour, there was a typo on this problem which made part (a) unsolvable. You were given full credit for (a) if you wrote down anything reasonable.]

(a) Let P be the point where ℓ_1 and ℓ_2 intersect. What is the distance between P and the point $(7,5,2)$?

We first find the point P by setting the equations for ℓ_1 and ℓ_2 equal:

$$(1,0,1) + t(1,2,-1) = \ell_1(t) = \ell_2(s) = (6,-2,2) + s(-1,2,-1)$$

This gives the system of equations

$$\begin{cases} 1+t = 6-s \\ 2t = -2+2s \\ 1-t = 2-s \end{cases} \qquad \text{with solution} \qquad \begin{cases} t = 2 \\ s = 3 \end{cases}$$

Therefore the intersection point P occurs at

$$P = \ell_1(2) = (1,0,1) + 2(1,2,-1) = (3,4,-1) \qquad \text{[2 points]}$$

or equivalently at

$$P = \ell_2(3) = (6,-2,2) + 3(-1,2,-1) = (3,4,-1).$$

We then compute the distance between $P = (3,4,-1)$ and $(7,5,2)$:

$$\begin{aligned} \text{Distance} &= \|(3,4,-1) - (7,5,2)\| = \sqrt{(3-7)^2 + (4-5)^2 + (-1-2)^2} \\ &= \sqrt{16+1+9} = \sqrt{26}. \qquad \text{[2 points]} \end{aligned}$$

(b) Find a normal vector $\mathbf{n}$ to the plane spanned by ℓ_1 and ℓ_2.

The plane is spanned by the vectors $(1,2,-1)$ and $(-1,2,-1)$ that point in the directions of ℓ_1 and ℓ_2, respectively. By the right hand rule, the cross product of $(1,2,-1)$ and $(-1,2,-1)$ is orthogonal to the plane, i.e. gives us a normal vector:

$$\begin{aligned} \mathbf{n} = (1,2,-1) \times (-1,2,-1) &= \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & -1 \\ -1 & 2 & -1 \end{bmatrix} \\ &= \big(2\cdot(-1) - (-1)\cdot 2\big)\mathbf{i} - \big(1\cdot(-1) - (-1)\cdot(-1)\big)\mathbf{j} + \big(1\cdot 2 - 2\cdot(-1)\big)\mathbf{k} \\ &= 0\cdot\mathbf{i} + 2\mathbf{j} + 4\mathbf{k} = (0,2,4). \qquad \text{[4 points]} \end{aligned}$$

(Notice that any scalar multiple of $\mathbf{n}$ is a normal vector to the plane, so there are many valid solutions to this problem.)

4. (a) Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be a function of two variables. Write down the definition of the partial derivatives of f:

$$\frac{\partial f}{\partial x}(x,y) = \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h}$$

[1 point]

$$\frac{\partial f}{\partial y}(x,y) = \lim_{h\to 0} \frac{f(x,y+h)-f(x,y)}{h}$$

(b) Consider the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

Using your answer from (a), calculate $\dfrac{\partial f}{\partial x}(0,0)$ and $\dfrac{\partial f}{\partial y}(0,0)$.

This is a straightforward limit calculation:

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{f(0+h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{h\cdot 0}{h^2+0^2} = 0$$

and

$$\frac{\partial f}{\partial y}(0,0) = \lim_{h\to 0} \frac{f(0,0+h)-f(0,0)}{h} = \lim_{h\to 0} \frac{0\cdot h}{0^2+h^2} = 0.$$

[3 points]

(c) Is the function $f(x,y)$ from part (b) differentiable at $(0,0)$?

No, the function f is not differentiable at $(0,0)$. [4 points — Of course, you must have justified your answer to receive any credit on this problem!]

One way to see this is to show that f is not continuous at $(0,0)$, hence cannot be differentiable. Indeed, the limit of $f(x,y)$ as (x,y) approaches $(0,0)$ along the x-axis is:

$$\lim_{\substack{y=0\\x\to 0}} f(x,y) = \lim_{x\to 0} \frac{x\cdot 0}{x^2+0^2} = 0,$$

while the limit of $f(x,y)$ as (x,y) approaches $(0,0)$ along the line $y=x$ is:

$$\lim_{\substack{y=x\\x\to 0}} f(x,y) = \lim_{x\to 0} \frac{x^2}{x^2+x^2} = \lim_{x\to 0} \frac{x^2}{2x^2} = \frac{1}{2}.$$

Since these limits do not agree, the limit

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$$

of $f(x,y)$ as (x,y) approaches $(0,0)$ does not exist. In particular, it cannot equal $f(0,0)=0$, so f is not continuous at $(0,0)$.

Another way to show that f is not differentiable at $(0,0)$ is to consider the limit in part (2) of the definition of differentiability:

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y) - \left(f(0,0) + \frac{\partial f}{\partial x}(0,0)\cdot(x-0) + \frac{\partial f}{\partial y}(0,0)\cdot(y-0)\right)}{\|(x,y)\|}$$

$$= \lim_{(x,y)\to(0,0)} \frac{f(x,y)}{\|(x,y)\|} = \lim_{(x,y)\to(0,0)} \frac{xy}{(x^2+y^2)^{3/2}}.$$

As (x,y) approaches $(0,0)$ along the x-axis, we find that

$$\lim_{\substack{y=0\\x\to 0}} \frac{xy}{(x^2+y^2)^{3/2}} = \lim_{x\to 0} \frac{x\cdot 0}{(x^2+0^2)^{3/2}} = 0,$$

while the limit as (x,y) approaches $(0,0)$ along the line $y=x$ does not exist:

$$\lim_{\substack{y=x\\x\to 0}} \frac{xy}{(x^2+y^2)^{3/2}} = \lim_{x\to 0} \frac{x^2}{(2x^2)^{3/2}} = \lim_{x\to 0} \frac{1}{2^{3/2}\cdot x} \longrightarrow \infty.$$

Therefore the limit in the definition of differentiability does not exist, and in particular is not 0.

(Another way to evaluate the limit is to change to polar coordinates and let $r\to 0$. You again get something which does not converge.)

5. The equation

$$z = x^2 - 3xy + y^2$$

determines a surface S in $\mathbb{R}^3$. Find the equation for the tangent plane to S at the point $(2, -3, 31)$.

The surface S is the graph of the function $f(x, y) = x^2 - 3xy + y^2$. Notice that the point $(2, -3, 31)$ lies on the surface S because

$$f(2, -3) = (2)^2 - 3(2)(-3) + (-3)^2 = 31.$$

The tangent plane to the graph of f at a point (x_0, y_0, z_0), where $z_0 = f(x_0, y_0)$, is the graph of the best affine approximation $T_{(x_0,y_0)}f(x, y)$. In other words, the tangent plane is given by the equation:

$$z = f(x_0, y_0) + \frac{\partial f}{\partial x}(x_0, y_0) \cdot (x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0) \cdot (y - y_0)$$

To find this equation for the values $x_0 = 2$ and $y_0 = -3$, we compute:

$$\frac{\partial f}{\partial x} = 2x - 3y \qquad \frac{\partial f}{\partial x}(2, -3) = 13$$

and

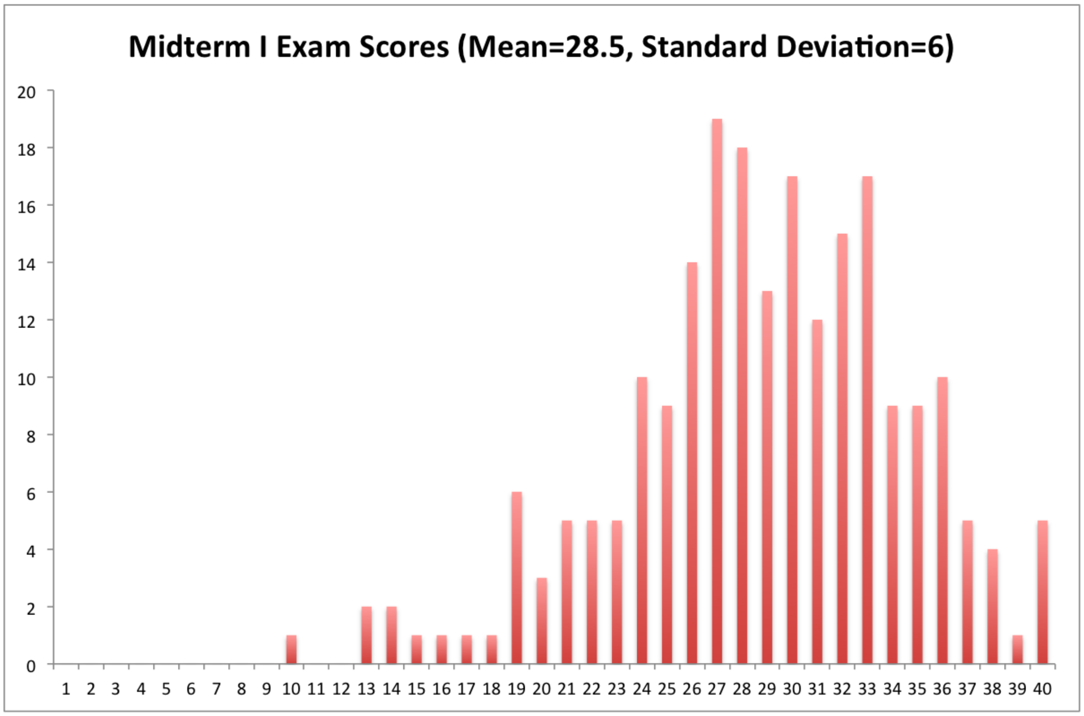
$$\frac{\partial f}{\partial y} = -3x + 2y \qquad \frac{\partial f}{\partial y}(2, -3) = -12$$

Therefore the equation for the tangent plane is:

$$z = 31 + 13(x - 2) - 12(y + 3).$$

Rearranging, we can write this slightly more cleanly as:

$$z = 13x - 12y - 31. \qquad \text{[8 points]}$$


Midterm I Exam Scores (Mean=28.5, Standard Deviation=6)
0
2
4
6
8
10
12
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1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31 32 33 34 35 36 37 38 39 40