MATH 202 — MIDTERM I

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1.(a) Give a unit vector normal to the plane $x - 2y + 2z + 5 = 0$ in $\mathbb{R}^3$.
Answer. A normal vector can be read from the equation of the plane: $\mathbf{n} = [1, -2, 2] = \mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$.
To obtain a unit vector: $\dfrac{\mathbf{n}}{||\mathbf{n}||} = \dfrac{\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}}{\sqrt{9}} = (1/3)\mathbf{i} - (2/3)\mathbf{j} + (2/3)\mathbf{k}$.

(b) The point M lying on the line segment AB is such that $\frac{|AM|}{|BM|} = 4$ (here $|AM|$ is the length of the line segment AM). If $A = (1, 2, 3)$ and $B = (5, -2, -9)$, determine the coordinates of M.
Answer. Let O be the origin of the system of coordinates. $\overrightarrow{OM} = \overrightarrow{OA} + \overrightarrow{AM} = \overrightarrow{OA} + (4/5)\overrightarrow{AB} = [1, 2, 3] + (4/5)[4, -4, -12] = [21/5, -6/5, -33/5]$. So $M = (21/5, -6/5, -33/5)$.

2. A triangle has vertices $(1, 0, 0)$, $(1, 1, 1)$, $(0, -2, 3)$. Find its area.
Answer. This is a homework problem. Let A, B, C denote the three points.
Then: $area(ABC) = (1/2)\|\overrightarrow{AB} \times \overrightarrow{AC}\|$.

$$\overrightarrow{AB}\times\overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 1 \\ -1 & -2 & 3 \end{vmatrix} = \mathbf{i}\begin{vmatrix} 1 & 1 \\ -2 & 3 \end{vmatrix} - \mathbf{j}\begin{vmatrix} 0 & 1 \\ -1 & 3 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 0 & 1 \\ -1 & -2 \end{vmatrix} = 5\mathbf{i} - \mathbf{j} + \mathbf{k}.$$

Hence $area(ABC) = (1/2)\|5\mathbf{i} - \mathbf{j} + \mathbf{k}\| = \frac{\sqrt{27}}{2} = \frac{3\sqrt{3}}{2}$.

3. Give the equation of a (continuous) path of non-constant velocity that traces out the (straight) line segment joining $A = (-1, 0, 1)$ and $B = (2, 3, 0)$. Verify your answer.
Answer. We discussed this problem in lecture (02/22/08). Let $\mathbf{c}(t)$ denote the path in question. One possible way of doing that is: $\overrightarrow{A\mathbf{c}(t)} = t^2\overrightarrow{AB}$, $0 \le t \le 1$.
This gives: $\mathbf{c}(t) = [-1, 0, 1] + t^2[3, 3, -1] = [3t^2 - 1, 3t^2, -t^2 + 1]$.
Verification: $\mathbf{c}'(t) = [6t, 6t, -2t]$ which is not constant (it depends on time t).

4. For $h(x, y) = f(\frac{x+y}{x-y})$, compute $x\frac{\partial h}{\partial x} + y\frac{\partial h}{\partial y}$.
Answer. If you want to apply the chain rule proper: $h(x, y) = f(u(x, y))$, with $u(x, y) = \frac{x+y}{x-y}$. Then: $D_h(x, y) = D_f(u(x, y)) * D_u(x, y)$ means:
$\nabla h(x, y) = f'(u(x, y)) * \nabla u(x, y)$.
Since $\frac{\partial u}{\partial x} = \frac{(x-y)-(x+y)}{(x-y)^2} = \frac{-2y}{(x-y)^2}$, and similarly $\frac{\partial u}{\partial y} = \frac{2x}{(x-y)^2}$, we get:
$\nabla h(x, y) = f'(u(x, y))[\frac{-2y}{(x-y)^2}, \frac{2x}{(x-y)^2}]$.
Finally: $x\frac{\partial h}{\partial x} + y\frac{\partial h}{\partial y} = f'(u(x, y))\left(\frac{-2xy}{(x-y)^2} + \frac{2xy}{(x-y)^2}\right) = 0$.

5. Determine the equation of the tangent plane at $(2, 1, 7)$ to the surface given by the equation $y = x^3 - yz$.
Answer. The surface in question is given by $x^3 - yz - y = 0$, so it is a level surface

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of $F(x, y, z) = x^3 - yz - y$.
A normal vector to the tangent plane is:
$\nabla F(2, 1, 7) = [3x^2, -z - 1, -y]\big|_{(2,1,7)} = [12, -8, -1]$.
The equation of the tangent plane is:
$12(X - 2) - 8(Y - 1) - (Z - 7) = 0$, i.e. $12X - 8Y - Z = 9$.

6. Consider the function of one variable $h(t) = f(t, t^2, t^3)$, where $f : \mathbb{R}^3 \to \mathbb{R}$ is a differentiable function such that $\dfrac{\partial f}{\partial x}(1, 1, 1) = 0$, $\dfrac{\partial f}{\partial y}(1, 1, 1) = a$, $\dfrac{\partial f}{\partial z}(1, 1, 1) = -1$.
Find a such that $h'(1) = 0$.
Answer. $h(t) = f(\mathbf{c}(t))$, where $\mathbf{c}(t) = (t, t^2, t^3)$.
The chain rule gives:
$h'(t) = \nabla f(\mathbf{c}(t)) \cdot \mathbf{c}'(t)$.
In particular, for $t = 1$ we have:
$h'(1) = \nabla f(\mathbf{c}(1)) \cdot \mathbf{c}'(1) = [0, a, -1] \cdot [1, 2, 3] = 2a - 3$.
$h'(1) = 0 \Rightarrow 2a - 3 = 0 \Rightarrow a = 3/2$.

Note. While it is true that $h'(t) = \nabla f(t, t^2, t^3) \cdot [1, 2t, 3t^2]$, it is not true that $h'(t) = 2at - 3t^2$. That's because the only information we have about f is at $(1, 1, 1)$, in other words it is **not** given that $\nabla f(t) = [0, -a, 1]$ for all t.