Mathematics 202: Midterm Exam 1 Name: SOLUTIONS

This exam consists of five questions, each worth eight points. For each question, give your response on the same page as the question is displayed, using the back side if necessary. Please show your work to receive full credit.

Circle your section below:

1. Abitbol T 1:30 Krieger 308
2. Guang T 3 NEB 12
3. Xiao Th 4:30 Krieger 304
4. McGonagle Th 3 Hodson 316
5. Xiao T 4:30 Krieger 308
6. Lorman Th 1:30 MD 309
7. Lorman Th 3 Shaffer 300
8. Biggs T 1:30 Latrobe 120
9. McGonagle T 3 Hodson 305

Problem #	Score
1	/ 8
2	/ 8
3	/ 8
4	/ 8
5	/ 8
TOTAL	/ 40

Do not turn this page until you are told to begin.

1. The line ℓ in $\mathbb{R}^3$ is parametrized by the function

$$\ell(t) = (2,1,2) + t(1,2,3).$$

Let P be the plane that contains the line ℓ and the point $(4,2,0)$.

(a) Find a normal vector $\mathbf{n}$ to the plane P.

Let us find two vectors $\mathbf{a}$ and $\mathbf{b}$ that are parallel to the plane P. Then their cross product $\mathbf{a} \times \mathbf{b}$ will be orthogonal to P, giving a normal vector.

The ℓ points in the direction of $(1,2,3)$, so $\mathbf{a} = (1,2,3)$ is parallel to P. Since the points $(2,1,2)$ and $(4,2,0)$ both lie on the plane P, their difference $\mathbf{b} = (4,2,0) - (2,1,2) = (2,1,-2)$ is also parallel to P.

Let $\mathbf{n} = \mathbf{a} \times \mathbf{b}$ is orthogonal to the plane spanned by $\mathbf{a}$ and $\mathbf{b}$, the cross product gives a normal vector to P. Explicitly,

$$\begin{aligned} \mathbf{n} = \mathbf{a} \times \mathbf{b} = (1,2,3) \times (2,1,-2) &= \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ 2 & 1 & -2 \end{bmatrix} \\ &= -7\mathbf{i} + 8\mathbf{j} - 3\mathbf{k} = (-7,8,-3). \end{aligned}$$

(Notice that there are *many* vectors orthogonal to P, so this problem has many solutions.)

(b) Write down the equation for the plane P.

A point (x,y,z) lies on the plane P if the difference vector

$$(x,y,z) - (2,1,2) = (x-2, y-1, z-2)$$

is parallel to P. A vector is parallel to P if and only if it is orthogonal to the normal vector $\mathbf{n}$, i.e. their inner product is zero. Therefore, we get the following equation for the plane P:

$$0 = (x-2, y-1, z-2) \cdot \mathbf{n} = (x-2, y-1, z-2) \cdot (-7, 8, -3).$$

Taking the dot product, this gives the equation:

$$-7(x-2) + 8(y-1) - 3(z-2) = 0.$$

2. Consider the function $f\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ defined by

$$f(x,y) = (x^3y - x, 3x^2y^2 + 2)$$

(a) Find the Jacobian matrix $Df_{(1,1)}$ of f at the point $(1,1)$.

The Jacobian matrix of f at (x,y) is the array of partial derivatives:

$$Df_{(x,y)} = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix} = \begin{bmatrix} 3x^2y - 1 & x^3 \\ 6xy^2 & 6x^2y \end{bmatrix}$$

Evaluated at $(x,y) = (1,1)$, this gives:

$$Df_{(1,1)} = \begin{bmatrix} 2 & 1 \\ 6 & 6 \end{bmatrix}$$

Recall that the best linear approximation L of f at $(1,1)$ is defined in terms of the Jacobian as:

$$L(x,y) = f(1,1) + Df_{(1,1)}(x-1, y-1).$$

(b) Evaluate $L(2,1)$.

First notice that the value of the linear function $Df_{(1,1)}$ at $(2-1, 1-1) = (1,0)$ is given by the Jacobian matrix:

$$Df_{(1,1)}(2-1, 1-1) = Df_{(1,1)}(1,0) = \begin{bmatrix} 2 & 1 \\ 6 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$$

(Although the matrix formalism writes this as a column vector, it is the same thing as the vector $(2,6) \in \mathbb{R}^2$.)

Evaluating the above formula for $L(x,y)$ at $(x,y) = (2,1)$, we get:

$$L(2,1) = f(1,1) + Df_{(1,1)}(2-1, 1-1) = (0,5) + (2,6) = (2,11).$$

(c) Let $g\colon \mathbb{R}^2 \to \mathbb{R}^3$ be defined by:

$$g(u, v) = (2u + v, ve^u, uv).$$

With f defined as in (a) and (b), compute the Jacobian matrix $D(g \circ f)_{(1,1)}$ using the chain rule.

The chain rule states that:

$$D(g \circ f)_{(1,1)} = Dg_{f(1,1)} \circ Df_{(1,1)}.$$

We know from (a) that

$$Df_{(1,1)} = \begin{bmatrix} 2 & 1 \\ 6 & 6 \end{bmatrix}.$$

Now we need to find the Jacobian of g at $f(1, 1) = (0, 5)$. In general the Jacobian is

$$Dg_{(u,v)} = \begin{bmatrix} 2 & 1 \\ ve^u & e^u \\ v & u \end{bmatrix}, \qquad \text{so } Dg_{f(1,1)} = Dg_{(0,5)} = \begin{bmatrix} 2 & 1 \\ 5 & 1 \\ 5 & 0 \end{bmatrix}$$

Therefore, the Jacobian of the composite is:

$$\begin{aligned} D(g \circ f)_{(1,1)} = Dg_{f(1,1)} \circ Df_{(1,1)} &= \begin{bmatrix} 2 & 1 \\ 5 & 1 \\ 5 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 6 & 6 \end{bmatrix} \\ &= \begin{bmatrix} 10 & 8 \\ 16 & 11 \\ 10 & 5 \end{bmatrix} \end{aligned}$$

3. (a) Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be a function of two variables. Write down the definition of the partial derivative $\dfrac{\partial f}{\partial x}$ of f with respect to x:

$$\frac{\partial f}{\partial x}(x,y) = \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h} \quad \text{(if it exists)}$$

(b) Consider the function f defined by

$$f(x,y) = \begin{cases} y & \text{if } y>0 \\ x^2 & \text{if } y=0 \\ 0 & \text{if } y<0 \end{cases}$$

Using your answer from (a), show that $\dfrac{\partial f}{\partial x}(x,y)$ exists for every point (x,y).

To show that $\dfrac{\partial f}{\partial x}(x,y)$ exists at (x,y), we evaluate the limit defining the partial derivative at (x,y). This requires three separate arguments depending on the sign of y.

If $y>0$, the partial derivative is:

$$\frac{\partial f}{\partial x}(x,y) = \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h} = \lim_{h\to 0} \frac{y-y}{h} = \lim_{h\to 0} \frac{0}{h} = 0.$$

If $y=0$, the partial derivative is:

$$\begin{aligned}\frac{\partial f}{\partial x}(x,y) = \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h} &= \lim_{h\to 0} \frac{(x+h)^2-x^2}{h} \\ &= \lim_{h\to 0} \frac{2xh+h^2}{h} = \lim_{h\to 0}(2x+h) = 2x.\end{aligned}$$

If $y<0$, the partial derivative is:

$$\frac{\partial f}{\partial x}(x,y) = \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h} = \lim_{h\to 0} \frac{y-y}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0.$$

In every case, the partial derivative exists.

4. Consider the curve in $\mathbb{R}^3$ defined by the function:

$$c(t) = (-t^2 + 5t)\mathbf{i} + (4\sqrt{t})\mathbf{j} + (t-1)\mathbf{k}.$$

(a) Find the velocity vector $\mathbf{v}$ of the curve at $t = 2$.

The velocity vector at time t is given by the derivative:

$$c'(t) = (-2t + 5)\mathbf{i} + (\frac{2}{\sqrt{t}})\mathbf{j} + \mathbf{k}.$$

At $t = 2$, the velocity vector is:

$$\mathbf{v} = c'(2) = \mathbf{i} + \sqrt{2}\mathbf{j} + \mathbf{k} = (1, \sqrt{2}, 1).$$

(b) What angle does the velocity vector $\mathbf{v}$ from part (a) make with the x-axis?

The fundamental theorem of the inner product states that the angle θ between two vectors $\mathbf{a}$ and $\mathbf{b}$ is given by the equation

$$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\|\|\mathbf{b}\| \cos\theta.$$

Let us apply this theorem to the vectors $\mathbf{v}$ and $\mathbf{i} = (1, 0, 0)$. Then the angle θ between $\mathbf{v}$ and the x-axis is given by the equation:

$$(1\sqrt{2}, 1) \cdot (1, 0, 0) = \|(1, \sqrt{2}, 1)\|\|(1, 0, 0)\| \cos\theta.$$

Evaluating both sides, we get:

$$1 = \sqrt{1 + 2 + 1} \cdot \sqrt{1} \cdot \cos\theta = 2\cos\theta.$$

Therefore, $\cos\theta = 1/2$, so $\theta = \pi/3$.

5. Let $f\colon \mathbb{R}^2 \longrightarrow \mathbb{R}$ be the function given by:

$$f(x, y, z) = 2x^2 + y^2 + z^3$$

(a) Compute the gradient of f:

$$\nabla f(x, y, z) = (4x, 2y, 3z^2)$$

(b) The equation $2x^2 + y^2 + z^3 = 14$ determines a surface S in $\mathbb{R}^2$. Find the equation for the tangent plane to S at $(1, 2, 2)$.

The surface S is a level set of the function f, so the tangent plane to S at $(1, 2, 2)$ is orthogonal to the vector $\nabla f(1, 2, 2)$. A point (x, y, z) is on the tangent plane if and only if the difference $(x, y, z) - (1, 2, 2) = (x - 1, y - 2, z - 2)$ is parallel to the tangent plane. This in turn is true if and only if the inner product with the normal vector $\nabla f(1, 2, 2)$ is zero:

$$(x - 1, y - 2, z - 2) \cdot \nabla f(1, 2, 2) = 0.$$

By (a), $\nabla f(1, 2, 2) = (4, 4, 12)$, so we get:

$$(x - 1, y - 2, z - 2) \cdot (4, 4, 12) = 0.$$

Taking the inner product, our equation is:

$$4(x - 1) + 4(y - 2) + 12(z - 2) = 0.$$