Vector Calculus   Sample Final Examination #1

Warning to Instructors: Question 2 may involve more linear algebra than you are assuming, so modify it accordingly (eg, by deleting or changing parts (b) and (c).

1. Let $f(x,y) = e^{xy}\sin(x+y)$.

   (a) In what direction, starting at $(0,\pi/2)$, is f changing the fastest?

   (b) In what directions starting at $(0,\pi/2)$ is f changing at 50% of its maximum rate?

   (c) Let $\mathbf{c}(t)$ be a flow line of $\mathbf{F} = \nabla f$ with $\mathbf{c}(0) = (0,\pi/2)$. Calculate

$$\frac{d}{dt}[f(c(t))]\bigg|_{t=0}.$$

2. Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be a given mapping and write $f(x,y,z) = (u(x,y,z), v(x,y,z), w(x,y,z))$. Let $g : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $g(u,v,w) = (u-v, u+w, w+v)$ and let $h = g \circ f$.

   (a) Write a formula for the derivative matrix $\mathbf{D}h$.

   (b) Show that $\mathbf{D}h$ cannot have rank 3 at any point (x,y,z).

   (c) Show that $\mathbf{D}h$ has an eigenvalue zero at every (x,y,z).

3. Extremize $f(x,y,z) = x$ subject to the constraints

$$x^2 + y^2 + z^2 = 1 \quad \text{and} \quad x + y + z = 1.$$

4. (a) Evaluate

$$\iiint_D \exp[(x^2+y^2+z^2)^{3/2}]\,dx\,dy\,dz$$

   where D is the region defined by $1 \le x^2+y^2+z^2 \le 2$ and $z \ge 0$.

   (b) Sketch or describe the region of integration for

$$\int_0^1 \int_0^x \int_0^y f(x,y,z)dz\,dy\,dx,$$

   and interchange the order to $dy\,dx\,dz$.

5. Let $\mathbf{G}(x,y) = (xe^{x^2+y^2} + 2xy)\mathbf{i} + (ye^{x^2+y^2} + x^2)\mathbf{j}$.

   (a) Show that $\mathbf{G} = \nabla f$ for some f; find such an f.

   (b) Use (a) to show that the line integral of $\mathbf{G}$ around the edge of the triangle with vertices $(0,0), (0,1), (1,0)$ is zero.

   (c) State Green's theorem for the triangle in (b) and a vector field $\mathbf{F}$ and verify it for the vector field $\mathbf{G}$ above.

6. Let W be the three dimensional region under the graph of $f(x, y) = \exp(x^2 + y^2)$ and over the region in the plane defined by $1 \leq x^2 + y^2 \leq 2$.

   (a) Find the volume of W.

   (b) Find the flux of the vector field $\mathbf{F} = (2x - xy)\mathbf{i} - y\mathbf{j} + yz\mathbf{k}$ out of the region W.

7. Let C be the curve $x^2 + y^2 = 1$ lying in the plane $z = 1$. Let $\mathbf{F} = (z - y)\mathbf{i} + y\mathbf{k}$.

   (a) Calculate $\nabla \times \mathbf{F}$.

   (b) Calculate $\int_C \mathbf{F} \cdot d\mathbf{s}$ using a parametrization of C and a chosen orientation for C.

   (c) Write $C = \partial S$ for a suitably chosen surface S and, applying Stokes' theorem, verify your answer in (b) .

   (d) Consider the sphere with radius $\sqrt{2}$ and center the origin. Let S' be the part of the sphere that is above the curve (*i.e.*, lies in the region $z \geq 1$), and has C as boundary. Evaluate the surface integral of $\nabla \times \mathbf{F}$ over S'. Specify the orientation you are using for S'.