COMPREHENSIVE TEST II FOR CHAPTERS 1–8

1. Choose the best answer to the following question. For any vectors $\mathbf{u}$ and $\mathbf{v}$, $\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) =$
   (i) $\mathbf{v} \cdot (\mathbf{u} \times \mathbf{v})$

   (ii) 0

   (iii) $(\mathbf{v} \times \mathbf{u}) \cdot \mathbf{u}$

   (a) (i) only (b) (ii) only (c) (i) and (ii) (d) (i), (ii) and (iii).

2. Suppose $\mathbf{u}$ and $\mathbf{v}$ are non-zero vectors in $\mathbb{R}^3$.
   (a) Suppose $\mathbf{u} \cdot \mathbf{v} = 0$. What is the geometric relation between $\mathbf{u}$ and $\mathbf{v}$?

   (b) Suppose $\mathbf{u} \times \mathbf{v} = \mathbf{0}$. What is the geometric relation between $\mathbf{u}$ and $\mathbf{v}$?

   (c) Suppose $\mathbf{u} \cdot \mathbf{v} = ||\mathbf{u}||\,||\mathbf{v}||$. What is the geometric relation between $\mathbf{u}$ and $\mathbf{v}$?

   (d) Suppose $||\mathbf{u} \times \mathbf{v}|| = ||\mathbf{u}||\,||\mathbf{v}||$. What is the geometric relation between $\mathbf{u}$ and $\mathbf{v}$?

3. Let $f(x, y, z) = x^2y + z$. At the point $(1, 1, 1)$, is f increasing faster in the direction of $(3, 0, 4)$ or in the direction of $(5, 12, 0)$? Explain.

4. (a) The cylinder $x^2 + y^2 = 4$ is cut by the plane $x + y + z = 1$. Show that the arc length of the intersecting curve is

$$\sqrt{8} \int_0^{2\pi} \sqrt{1 - \cos\theta \sin\theta}\, d\theta.$$

   (b) For the intersecting curve, find an equation for the tangent line at $(1, \sqrt{3}, -\sqrt{3})$.

5. Find the absolute maximum and minimum values of $f(x, y) = e^x + 5$ on the circle $x^2 + y^2 = 4$.

6. Let S be the boundary of a box $B = [-2, 2] \times [-1, 1] \times [-3, 3]$, $\mathbf{F}(x, y, z) = 2x\mathbf{i} + 3z\mathbf{j} + 2y\mathbf{k}$, and $\mathbf{G}(x, y, z) = x^3\mathbf{i} + 3z\mathbf{j} + 2y\mathbf{k}$.
   (a) Compute the integral of $\nabla \cdot \mathbf{F}$ over B.

(b) Compute $\iint_S \mathbf{G} \cdot d\mathbf{S}$.

(c) Suppose the origin at the center of B is shifted to $(8, -15, 20)$ and then rotated 30° around the y axis. Compute $\iint_S \mathbf{F} \cdot d\mathbf{S}$.

7. A hole of radius $1/2$ is drilled through the axis of symmetry of the hemisphere $x^2 + y^2 + z^2 = 1$, $z \geq 0$.
   (a) Write the volume of the remaining piece in Cartesian coordinates.

   (b) Write the volume of the remaining piece in cylindrical coordinates.

   (c) Compute the volume.

8. Compute $\int_0^1 \int_x^1 \cos(y^2 + 3)dy\, dx$.

9. (a) Verify Stokes' theorem for $\mathbf{F} = z^3\mathbf{i} + (x^3 - y^3)\mathbf{j} + y^3\mathbf{k}$ over the hemisphere $x^2 + y^2 + z^2$, with $z \geq 0$.

   (b) For the same $\mathbf{F}$ as in (a), evaluate $\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}$ for the surface $x^2 + y^2 + 5z^2$, with $z \leq 0$.

10. (a) Find a vector-valued function $f(x, y, z)$ such that

$$\mathbf{D}f(x, y, z) = \begin{bmatrix} -yz\sin(xy)e^{\cos(xy)} & -xz\sin(xy)e^{\cos(xy)} & e^{\cos(xy)} \\ y^2 \sin z & 2xy\sin z & xy^2 \cos z \end{bmatrix}.$$

   (b) For the region D shown in Figure 1, let V be the volume of the solid lying between $f(x, y) = x^3 \sin y$ and the xy plane and lying over D. Write V in the form $\iiint g(x, y, z)\, dz\, dy\, dx$.

   (c) Rewrite your answer to part (b) in the form $\iiint g(x, y, z)\, dz\, dx\, dy$.

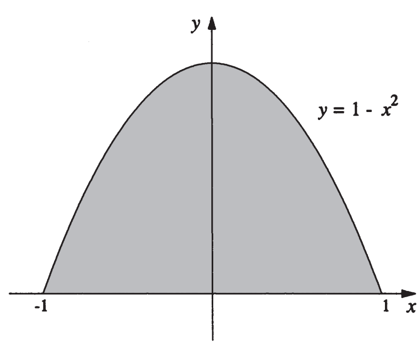

FIGURE 1