Mathematics 202 NAME ______________________

Practice Final Examination

April 30, 2009

This exam consists of **10** problems, numbered **1–10**. For partial credit **you must present your work clearly and understandably; no credit will be given for unsupported answers.**

If more space than allotted is needed, use the back of the previous page and make note of this. Please make your final answer clear by circling it. **Except for your name, please refrain from writing anywhere else on this page.**

During the exam, no calculators, computers, or other electronic aids are allowed, nor are you allowed to refer to any written notes or source material, nor to communicate with other students.
Please switch off all mobile phones

Check this examination booklet before you start. There should be 10 problems on 16 pages (including this one).

Question	**Points**	**Score**
1	20	
2	15	
3	20	
4	30	
5	20	
6	10	
7	20	
8	30	
9	15	
10	20	
Total	200	

1. [20 points]

Answer **TRUE** or **FALSE** to the following questions. Each question is worth 4 points. You do not need to provide an explanation of your answer.

(a) If $f : \mathbb{R}^3 \to \mathbb{R}$ is C^2 then $\nabla \cdot \nabla f = 0$.

FALSE.

$$\nabla \cdot \nabla f = \nabla \cdot \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

which may be non-zero.

Example: $f(x,y,z) = x^2 + y^2 + z^2$.

(b) If $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ is C^2 then $\nabla \cdot (\nabla \times \mathbf{F}) = 0$.

TRUE

One can prove this by explicit calculation.

(c) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a C^2 function with a critical point at (x_0, y_0). If the determinant of the Hessian of f at (x_0, y_0) is negative then f has a local maximum at (x_0, y_0).

FALSE.

f has a saddle point if the determinant of the Hessian is negative.

(d) A constant vector field is conservative.

TRUE

A vector field $\vec{F}(x,y,z) = (a,b,c)$ satisfies $\vec{F} = \nabla f$, where $f(x,y,z) = ax + by + cz$.

(e) The iterated integral

$$\int_0^{2\pi} \int_0^2 \int_{2r}^4 r \, dz \, dr \, d\theta$$

represents the volume enclosed by the cone of height 4 and radius 2.

TRUE

Sketch the cone and see for yourself!

2. [15 points]

Compute the second-order Taylor expansion of the function

$$f(x,y) = \sin(xy) + \cos(xy)$$

about the point $(x_0, y_0) = (\frac{1}{4}\pi, 1)$.

$$f\left(\frac{\pi}{4}, 1\right) = \sin\frac{\pi}{4} + \cos\left(\frac{\pi}{4}\right) = \sqrt{2}$$

$$\frac{\partial f}{\partial x} = y\cos(xy) - y\sin(xy) \qquad \frac{\partial f}{\partial x}\left(\frac{\pi}{4}, 1\right) = \cos\left(\frac{\pi}{4}\right) - \sin\left(\frac{\pi}{4}\right) = 0$$

$$\frac{\partial f}{\partial y} = x\cos(xy) - x\sin(xy) \qquad \frac{\partial f}{\partial y}\left(\frac{\pi}{4}, 1\right) = \frac{\pi}{4}\left(\cos\left(\frac{\pi}{4}\right) - \sin\left(\frac{\pi}{4}\right)\right) = 0$$

$$\frac{\partial^2 f}{\partial x^2} = -y^2\sin(xy) - y^2\cos(xy) \qquad \frac{\partial^2 f}{\partial x^2}\left(\frac{\pi}{4}, 1\right) = -\sin\left(\frac{\pi}{4}\right) - \cos\left(\frac{\pi}{4}\right) = -\sqrt{2}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \cos(xy) - xy\sin(xy) - \sin(xy) - xy\cos(xy)$$

$$\frac{\partial^2 f}{\partial y^2} = -x^2\sin(xy) - x^2\cos(xy)$$

$$\frac{\partial^2 f}{\partial x \partial y}\left(\frac{\pi}{4}, 1\right) = \cos\left(\frac{\pi}{4}\right) - \sin\left(\frac{\pi}{4}\right) - \frac{\pi}{4}\sin\left(\frac{\pi}{4}\right) - \frac{\pi}{4}\cos\left(\frac{\pi}{4}\right) = -\frac{\pi}{4}\sqrt{2}$$

$$\frac{\partial^2 f}{\partial y^2} = -\left(\frac{\pi}{4}\right)^2\sin\left(\frac{\pi}{4}\right) - \left(\frac{\pi}{4}\right)^2\cos\left(\frac{\pi}{4}\right) = -\left(\frac{\pi}{4}\right)^2\sqrt{2}$$

$$f(x,y) = \sqrt{2} + \frac{1}{2}(-\sqrt{2})\left(x - \frac{\pi}{4}\right)^2 + \left(-\frac{\pi}{4}\sqrt{2}\right)\left(x - \frac{\pi}{4}\right)(y-1) + \frac{1}{2}\left(-\left(\frac{\pi}{4}\right)^2\sqrt{2}\right)(y-1)^2 + R_2(x,y)$$

$$= \sqrt{2} - \frac{\sqrt{2}}{2}\left(x - \frac{\pi}{4}\right)^2 - \frac{\pi}{4}\sqrt{2}\left(x - \frac{\pi}{4}\right)(y-1) - \left(\frac{\pi}{4}\right)^2\frac{\sqrt{2}}{2}(y-1)^2 + R_2(x,y)$$

3. [20 points]

(a) (5 points)

State Green's theorem.

D a region in $\mathbb{R}^2$

C is the positively oriented boundary of D

$P, Q : D \to \mathbb{R}$ C^1 functions

$\vec{F}(x,y) = (P, Q)$.

Then

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \int_C \vec{F} \cdot d\vec{s}.$$

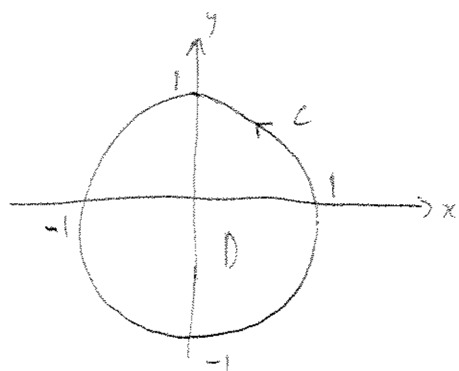
(b) (15 points)

Let C be the unit circle in the xy plane oriented anti-clockwise, and let

$$\mathbf{F} = \left(-y^3 + \sin(\sin x), x^3 + \sin(\sin y)\right)$$

Use Green's theorem to compute $\int_C \mathbf{F} \cdot \mathbf{ds}$.

$P(x,y) = -y^3 + \sin(\sin x)$, $Q(x,y) = x^3 + \sin(\sin y)$.



$$\frac{\partial Q}{\partial x} = 3x^2$$

$$\frac{\partial P}{\partial y} = -3y^2$$

Green's theorem: $\int_C \vec{F} \cdot d\vec{s} = \iint_D 3x^2 + 3y^2 \, dx\,dy$

$$= \int_0^{2\pi} \int_0^1 3r^2 \, r \, dr \, d\theta$$

$$= \int_0^{2\pi} \left[\frac{3}{4} r^4\right]_0^1 d\theta$$

$$= \int_0^{2\pi} \frac{3}{4} \, d\theta = \frac{3\pi}{2}$$

So $\int_C \vec{F} \cdot d\vec{s} = \frac{3\pi}{2}$.

4. [30 points]

(a) (5 points)

State Stokes' theorem.

S an oriented surface in $\mathbb{R}^3$

C positively oriented boundary curve

$\vec{F} : S \to \mathbb{R}^3$ a C^1 vector field.

Then $$\iint_S (\nabla \times \vec{F}) \cdot d\vec{S} = \int_C \vec{F} \cdot d\vec{s}$$

(b) (25 points)

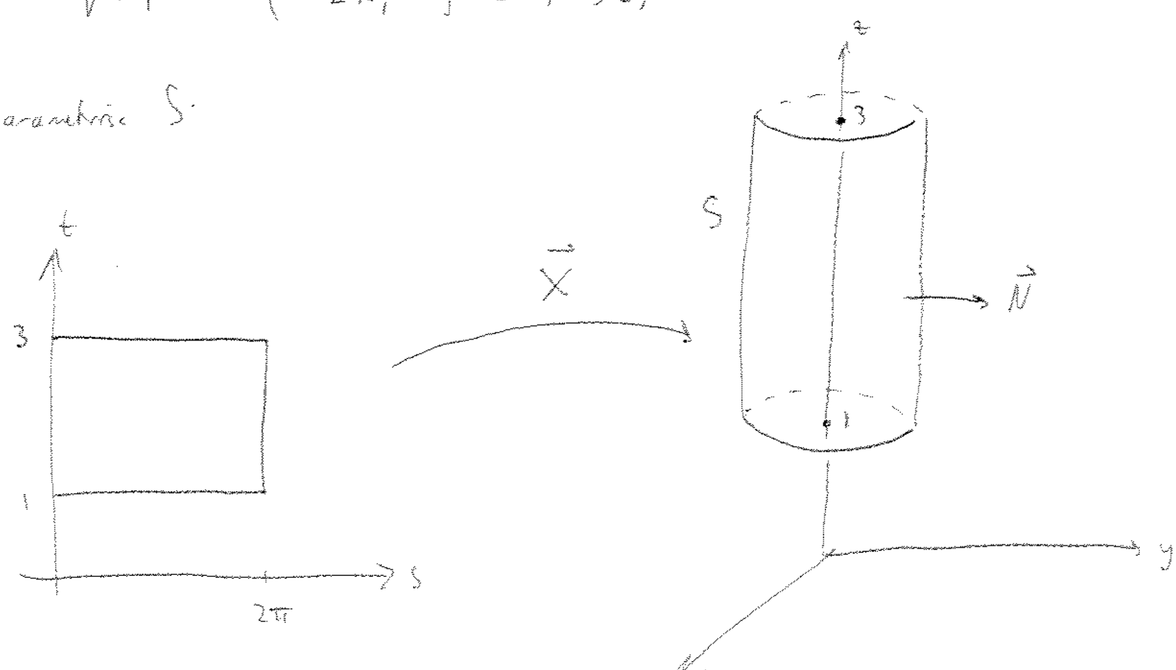
Verify Stokes' theorem for the case where $S = \{(x, y, z) : x^2 + y^2 = 25, 1 \le z \le 3\}$ is the curved surface of a cylinder with outwards pointing normal vector, and $\mathbf{F}$ is the vector field $\mathbf{F}(x, y, z) = (-zy, 2zx, x^2)$.

Note: To get full credit for this question you must compute both sides of Stokes' theorem and show that they are equal.

First compute the left-hand side of Stokes' theorem: $\iint_S \nabla \times \vec{F} \cdot d\vec{S}$.

$$\nabla \times \vec{F} = (-2x, -y-2x, 3z)$$

Parametrise S:



(Extra space for problem 4)

$$\vec{X}(s,t) = (5\cos s,\ 5\sin s,\ t) \qquad (s,t) \in [0,2\pi] \times [1,3].$$

$$\vec{T}_s = \frac{\partial \vec{X}}{\partial s} = (-5\sin s,\ 5\cos s,\ 0)$$

$$\vec{T}_t = \frac{\partial \vec{X}}{\partial t} = (\ 0\ ,\ 0\ ,\ 1)$$

$$\vec{T}_s \times \vec{T}_t = (5\cos s,\ 5\sin s,\ 0)$$

(Check: this is the <u>outwards</u> pointing normal vector)

$$\nabla \times \vec{F}\,(\vec{X}(s,t)) = (-10\cos s,\ -5\sin s - 10\cos s,\ 3t)$$

$$\iint_S (\nabla \times \vec{F}) \cdot d\vec{S} = \int_1^3 \int_0^{2\pi} (-10\cos s,\ -5\sin s - 10\cos s,\ 3t) \cdot (5\cos s,\ 5\sin s, 0)\, ds\, dt$$

$$= \int_1^3 \int_0^{2\pi} -50\cos^2 s - 25\sin^2 s - 50\sin s\cos s \ \ ds\, dt$$

$$= \int_1^3 \int_0^{2\pi} -25(1+\cos 2s) - \tfrac{25}{2}(1-\cos 2s) - 25\sin 2s\ ds\, dt$$

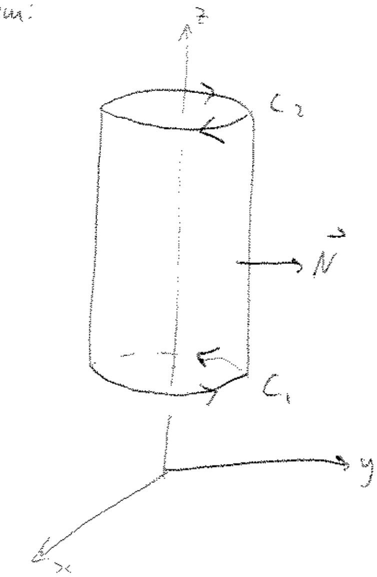
$$= \int_1^3 -75\pi\ dt$$

$$= -150\pi.$$

(Extra space for problem 4)

The right-hand side of Stokes' theorem:

$$\int_C \vec{F} \cdot d\vec{s} :$$



Positive orientation on C_1 and C_2.

Parametrise C_1:

$$\vec{c}_1(t) = (5\cos t, 5\sin t, 1)$$

~~$\vec{c}_2(t) = (8\cos t, 8\sin t, 3)$~~

$$\vec{c}_1\,'(t) = (-5\sin t, 5\cos t, 0)$$

$$\vec{F}(\vec{c}_1(t)) = (-5\sin t, 10\cos t, 25\cos^2 t)$$

Parametrise C_2:

$$\vec{c}_2(t) = (5\cos t, -5\sin t, 3)$$

$$\vec{c}_2\,'(t) = (-5\sin t, -5\cos t, 0)$$

$$\vec{F}(\vec{c}_2(t)) = (15\sin t, 30\cos t, 25\cos^2 t)$$

$$\int_C \vec{F} \cdot d\vec{s} = \int_{C_1} \vec{F} \cdot d\vec{s} + \int_{C_2} \vec{F} \cdot d\vec{s}$$

$$= \int_0^{2\pi} (-5\sin t, 10\cos t, 25\cos^2 t) \cdot (-5\sin t, 5\cos t, 0)\, dt$$

$$+ \int_0^{2\pi} (15\sin t, 30\cos t, 25\cos^2 t) \cdot (-5\sin t, -5\cos t, 0)\, dt$$

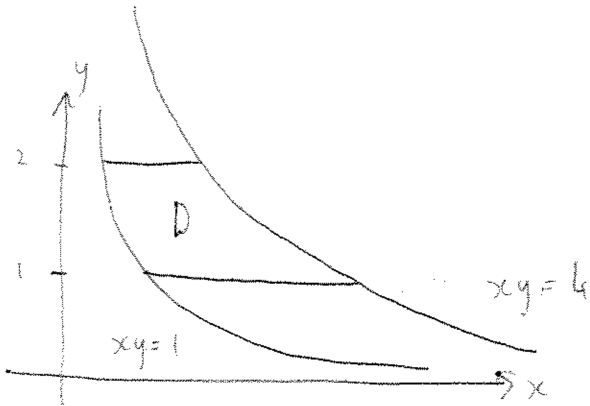
$$= \int_0^{2\pi} 25\sin^2 t + 50\cos^2 t \, dt + \int_0^{2\pi} -75\sin^2 t - 150\cos^2 t \, dt$$

$$= \cdots = 75\pi - 225\pi = -150\pi$$ which matches our answer for $\iint_S \nabla \times \vec{F} \cdot d\vec{S}$

5. [20 points]

Let D be the region bounded by $xy = 1$, $xy = 4$, $y = 1$ and $y = 2$.

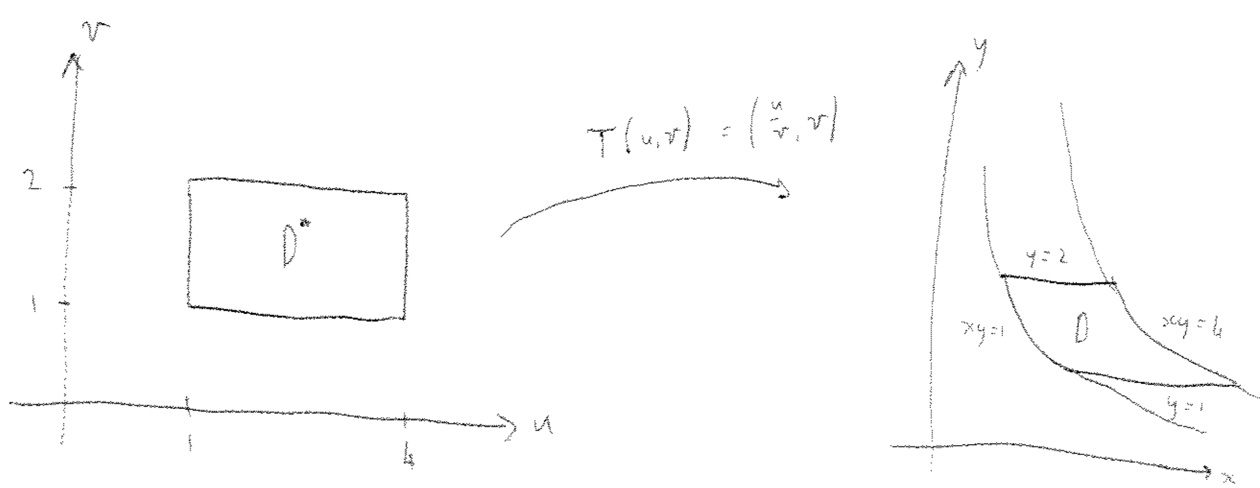
(a) (5 points) Sketch the region D.



(b) (5 points) Consider the co-ordinate change $u = xy$, $v = y$. Solve for (x, y) in terms of (u, v), compute the Jacobian $\frac{\partial(x,y)}{\partial(u,v)}$, and sketch the region in the uv-plane corresponding to the region D in the xy-plane.

$$x = \frac{u}{v}, \quad y = v$$

$$\Rightarrow \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{v} & -\frac{u}{v^2} \\ 0 & 1 \end{vmatrix} = \frac{1}{v}$$



(c) (10 points) Compute the double integral

$$\iint_D \frac{1}{x^2y^2+1}\,dxdy$$

$$\iint_D \frac{1}{x^2y^2+1}\,dxdy = \iint_{D^*} \frac{1}{u^2+1}\cdot\frac{1}{v}\,dudv$$

Jacobian

$$= \int_1^2\int_1^4 \frac{1}{u^2+1}\cdot\frac{1}{v}\,dudv$$

$$= \int_1^2 \left[\frac{1}{v}\arctan u\right]_1^4 dv$$

$$= \int_1^2 \frac{1}{v}(\arctan 4 - \arctan 1)\,dv$$

$$= \left[\log v\,(\arctan 4 - \arctan 1)\right]_1^2$$

$$= (\log 2 - \log 1)(\arctan 4 - \arctan 1)$$

$$= \log 2\,(\arctan 4 - \arctan 1)$$

"natural log"

base e.

6. [10 points] Let C be the curve given by

$$\mathbf{c}(t) = (2t, t^2, t^3/3) \quad \text{for } 0 \leq t \leq 1.$$

(a) Find the total arc length of C. (Hint: $(t^2+2)^2 = t^4 + 4t^2 + 4$.)

Arclength of $\vec{c}(t)$ for $a \le t \le b$ is

$$\int_0^b \|\vec{c}\,'(t)\|\, dt$$

In this case: $\vec{c}(t) = \left(2t, t^2, \frac{t^3}{3}\right) \qquad 0 \le t \le 1$

$$\vec{c}\,'(t) = (2, 2t, t^2)$$

$$\|\vec{c}\,'(t)\| = \sqrt{4 + 4t^2 + t^4}$$

$$= \sqrt{(2+t^2)^2}$$

$$= |2+t^2| = 2+t^2$$ (since it is always positive)

$\Rightarrow$ Arclength $= \int_0^1 2+t^2\, dt = \left[2t + \frac{1}{3}t^3\right]_0^1$

$$= 2 + \frac{1}{3} = \frac{7}{3}.$$

7. [20 points]

Use Lagrange multipliers to find the dimensions of a solid cylinder (i.e. find the radius and height) with fixed surface area A and maximum volume.

Note: The surface area of the solid cylinder is the surface area of the curved side, plus the surface area of the two ends.

r = radius

h = height

$f(r,h) = \pi r^2 h$ = volume

$g(r,h) = 2\pi r^2 + 2\pi r h$ = surface area. (constraint)

Lagrange's condition:

$$\nabla f = \lambda \nabla g$$

$$\Rightarrow (2\pi r h, \pi r^2) = \lambda(4\pi r + 2\pi h, 2\pi r)$$

$$\begin{cases} 2\pi r h = 4\pi \lambda r + 2\pi \lambda h & (1) \\ \pi r^2 = 2\pi \lambda r & (2) \end{cases}$$

From (2): $\lambda = \frac{1}{2} r$

Sub into (1): $rh = r^2 + \frac{1}{2} rh$

$\Rightarrow rh = 2r^2 \Rightarrow h = 2r$ or $r = 0$. Can ignore, as this is the case of an infinitely thin cylinder.

Sub into constraint equation:

$$2\pi r^2 + 4\pi r^2 = A$$

$$\Rightarrow \boxed{r = \sqrt{\frac{A}{6\pi}}} \quad \boxed{h = 2\sqrt{\frac{A}{6\pi}}}$$

8. [30 points]

Let B be the inverted paraboloid $B = \{(x, y, z) \in \mathbb{R}^3 : 0 \leq z \leq 4 - x^2 - y^2\}$, and let $\mathbf{F}(x, y, z) = (x, y, z - x^2)$.

(a) (5 points) State Gauss' theorem

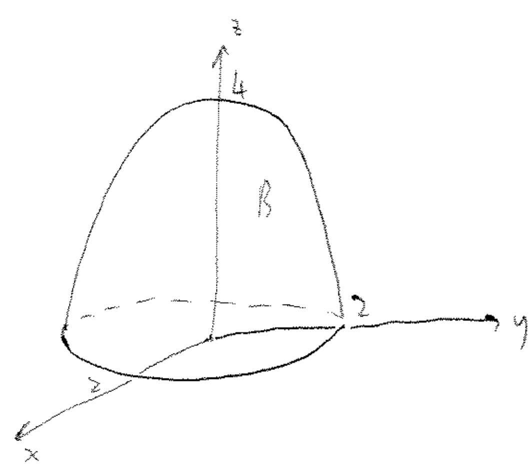
B a solid region in $\mathbb{R}^3$

S = boundary of B, oriented with normal vector pointing away from B.

$\vec{F}$ a C^1 vector field.

Then $\iiint_B \nabla \cdot \vec{F} \, dV = \iint_S \vec{F} \cdot d\vec{S}$.

(b) (5 points) Sketch the region B in $\mathbb{R}^3$.



(c) (20 points) Verify Gauss' theorem for this case.

Note: To get full credit for this question you must compute both sides of Gauss' theorem and show that they are equal.

First, compute $\iiint_B \nabla\cdot\vec{F}\, dV$

$$\nabla\cdot\vec{F} = 1+1+1 = 3$$

So $\iiint_B \nabla\cdot\vec{F}\, dV = \iint_D \left(\int_0^{4-x^2-y^2} 3\, dz\right) dx\,dy$ where D is the disk of radius 2 in the xy plane

$$= \int_0^{2\pi}\int_0^2\int_0^{4-r^2} 3r\,dz\,dr\,d\theta \quad \text{(cylindrical co-ordinates)}$$

$$= \int_0^{2\pi}\int_0^2 (4-r^2)\,3r\,dr\,d\theta$$

$$= \int_0^{2\pi} \left[6r^2 - \frac{3}{4}r^4\right]_0^2 d\theta$$

$$= \int_0^{2\pi} 24 - 12\,d\theta$$

$$= 24\pi.$$

(Extra space for problem 8)

The surface S has two components: S_1 (curved part of the paraboloid)

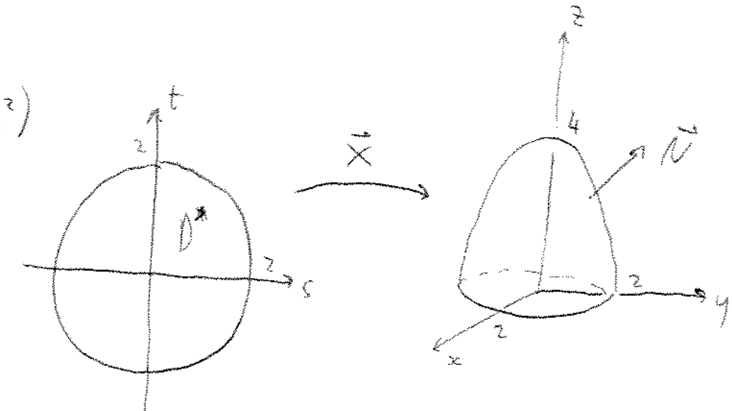
S_2 (disk of radius 2 in the xy plane)

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_{S_1} \vec{F}\cdot d\vec{S} + \iint_{S_2} \vec{F}\cdot d\vec{S}.$$

Parametrise S_1: $\vec{X}(s,t) = (s, t, 4-s^2-t^2)$

$$\vec{T}_s = \frac{\partial \vec{X}}{\partial s} = (1, 0, -2s)$$

$$\vec{T}_t = \frac{\partial \vec{X}}{\partial t} = (0, 1, -2t)$$



$\vec{T}_s \times \vec{T}_t = (2s, 2t, 1)$ (check orientation, it should point upwards).

$$\vec{F}(\vec{X}(s,t)) = (s, t, 4-s^2-t^2-s^2) = (s, t, 4-2s^2-t^2)$$

$$\Rightarrow \iint_{S_1} \vec{F}\cdot d\vec{S} = \iint_{D^*} (s, t, 4-2s^2-t^2)\cdot(2s, 2t, 1)\, ds\, dt$$

$$= \iint_{D^*} 2s^2 + 2t^2 + 4 - 2s^2 - t^2 \, ds\, dt$$

$$= \int_0^{2\pi}\int_0^2 (4 + r^2\sin^2\theta)\, r \, dr\, d\theta \quad \text{(change to polar co-ords)}$$

$$= \int_0^{2\pi} \left[2r^2 + \tfrac{1}{4} r^4 \sin^2\theta\right]_0^2 d\theta$$

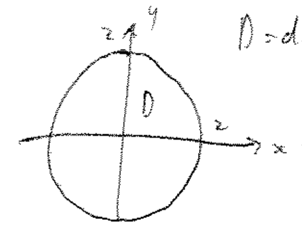
$$= \int_0^{2\pi} 8 + 2(1-\cos 2\theta)\, d\theta$$

$$= 16\pi + 4\pi = 20\pi$$

(Extra space for problem 8)

Parametrise S_2: (note that the normal must point <u>downwards</u> for Gauss' theorem).

$$\iint_{S_2} \vec{F}\cdot d\vec{S} = \iint_D (x, y, z - x^2)\cdot(0,0,-1)\, dx\,dy$$



D = disk in xy plane

$$= \iint_D x^2 - 0 \, dx\,dy$$

$z = 0$ on D

$$= \int_0^{2\pi}\int_0^2 r^2\cos^2\theta \; r\,dr\,d\theta \qquad \text{(change to polar co-ords)}$$

$$= \int_0^{2\pi} \left[\frac{1}{4} r^4 \cos^2\theta\right]_0^2 d\theta$$

$$= \int_0^{2\pi} 4\cdot\frac{1}{2}(1+\cos 2\theta)\, d\theta$$

$$= 4\pi.$$

So $\iint_S \vec{F}\cdot d\vec{S} = \iint_{S_1}\vec{F}\cdot d\vec{S} + \iint_{S_2}\vec{F}\cdot d\vec{S}$

$= 20\pi + 4\pi$

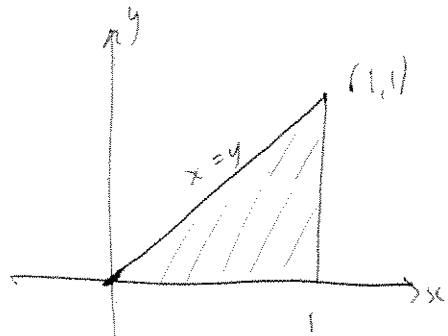
$= 24\pi$, the same answer as for $\iiint_B \nabla\cdot\vec{F}\,dV$.

9. [15 points]

This question concerns the following double integral.

$$\int_0^1 \left(\int_y^1 e^{x^2}\, dx \right) dy$$

(a) (5 points) Sketch the region of integration.



(b) (10 points) Evaluate the integral by integrating with respect to y first.

$$\int_0^1 \left(\int_y^1 e^{x^2} dx \right) dy = \int_0^1 \int_0^x e^{x^2}\, dy\, dx$$

$$= \int_0^1 x e^{x^2}\, dx$$

Sub $u = x^2$
$du = 2x\,dx$
$\frac{1}{2} du = x\,dx$

$$= \int_0^1 \frac{1}{2} e^u\, du$$

$$= \left[\frac{1}{2} e^u \right]_0^1$$

$$= \frac{1}{2}(e-1).$$

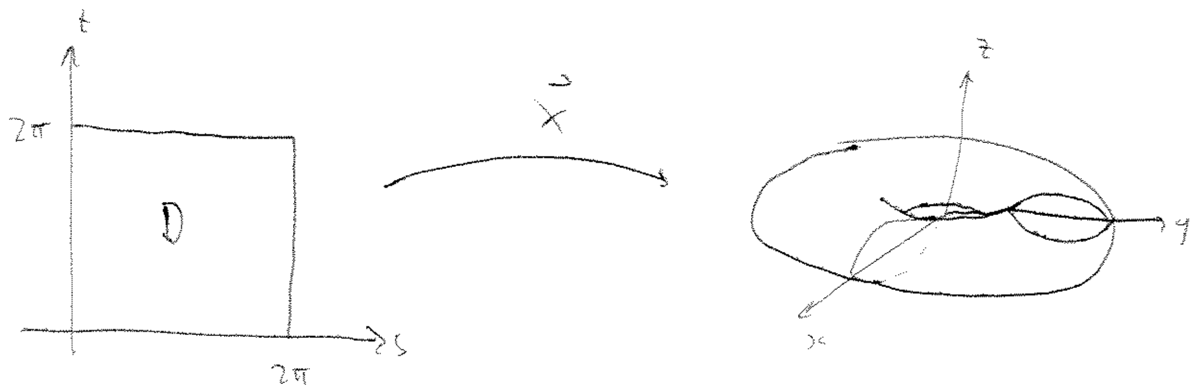
10. [20 points]

The torus T can be parametrised by the function $\mathbf{X} : D \to \mathbb{R}^3$, where

$$\mathbf{X}(s,t) = ((R + \cos s)\cos t, (R + \cos s)\sin t, \sin s),$$

$R > 1$ is fixed, and D is the rectangle $[0, 2\pi] \times [0, 2\pi]$.

Show that the surface area is $(2\pi)^2 R$.



$$\vec{T}_s = \frac{\partial \vec{X}}{\partial s} = \left(-\sin s \cos t, -\sin s \sin t, \cos s \right)$$

$$\vec{T}_t = \frac{\partial \vec{X}}{\partial t} = \left(-(R+\cos s)\sin t, (R+\cos s)\cos t, 0 \right)$$

$$\vec{T}_s \times \vec{T}_t = \left(-(R+\cos s)\cos t \cos s, -(R+\cos s)\sin t \cos s, -(R+\cos s)\sin s(\cos^2 t + \sin^2 t) \right)$$

$$\Rightarrow \|\vec{T}_s \times \vec{T}_t\| = \sqrt{(R+\cos s)^2\cos^2 t\cos^2 s + (R+\cos s)^2 \sin^2 t \cos^2 s + (R+\cos s)^2 \sin^2 s}$$

$$= \sqrt{(R+\cos s)^2 \cos^2 s + (R+\cos s)^2 \sin^2 s}$$

$$= |R+\cos s| = R+\cos s$$ since $R > 1$, and so $R+\cos s$ is always positive.

So surface area $= \iint_D \|\vec{T}_s \times \vec{T}_t\| \, ds\,dt = \int_0^{2\pi}\int_0^{2\pi} R+\cos s \, ds\,dt$

$$= (2\pi)^2 R.$$