# Math 202 – Calculus III

Practice Final

**Grading**

| | |
|---|---|
| 1 | 0 |
| 2 | 0 |
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| 4 | 0 |
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| 7 | 0 |
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| 9 | 0 |
| 10 | 0 |
| **Total:** | $\infty$ |

▶ **Your PRINTED name is:** **Practice Final** Solution

▶ **Please circle your section:**

| | | | |
|---|---|---|---|
| (1) | T 1:30 | Hodson 216 | McGonagle, Matthew |
| (2) | T 3:00 | Bloomberg 168 | McGonagle, Matthew |
| (3) | Th 4:30 | Krieger 308 | Lin, Longzhi |
| (4) | Th 1:30 | Shaffer 300 | Lin, Longzhi |
| (5) | Th 4:30 | Krieger 300 | Banerjee, Romie |
| (6) | Th 1:30 | Dunning 205 | Banerjee, Romie |
| (7) | Th 3:00 | Bloomberg 168 | Lin, Longzhi |
| (8) | Th 4:30 | Krieger 302 | Cutrone, Joseph |

▶ **Write out and SIGN the pledge:**

I pledge my honor that I have not violated the Honor Code during this examination.

Signature: Date:

▶ **This is a 3-hour closed book exam. This examination booklet contains 10 problems, including one bonus problem, on 13 sheets of paper including the front cover. Please detach the last two pages before exam, which is intended for use as scrap paper.**

**1 (30 pts.)** Which of the following statements are true? Put a (T) before the correct ones and an (F) before the wrong ones. (No reasoning is required.)

( F ) Any vector has a unique length and a unique direction.

( F ) If $\vec{a} \cdot \vec{b} = \vec{a} \cdot \vec{c}$, then $\vec{b} = \vec{c}$.

( T ) Any local extremal point of a smooth function on $\mathbb{R}^2$ is a critical point.

( T ) There is no smooth vector field $\vec{F}$ on $\mathbb{R}^3$ such that $\nabla \times \vec{F} = \langle x, y, z \rangle$.

( T ) The flux of a smooth planar vector field $\vec{F}$ out of the unit circle equals $\pi \cdot \mathrm{div}\vec{F}(P)$ for some point $P$ in the unit disc.

( F ) Any smooth surface has exactly two orientations.

**2 (20 pts.)** Let $A = (0,1,3), B = (2,1,-1), C = (0,3,2)$ be three points in $\mathbb{R}^3$.

(1) Find $\overrightarrow{AB} \cdot (\overrightarrow{BC} \times \overrightarrow{CA})$.

$$\left.\begin{array}{l} \overrightarrow{BC}, \overrightarrow{CA} \text{ sits in } \Pi \Rightarrow \overrightarrow{BC} \times \overrightarrow{CA} \perp \Pi \\ \qquad\qquad\qquad\qquad \overrightarrow{AB} \text{ sits in } \Pi \end{array}\right\} \Rightarrow \overrightarrow{AB} \cdot (\overrightarrow{BC} \times \overrightarrow{CA}) = 0.$$

(2) Find the area of the triangle $\triangle ABC$.

$$\overrightarrow{BC} \times \overrightarrow{CA} = \langle -2, 2, 3 \rangle \times \langle 0, -2, 1 \rangle = \langle 8, 2, 4 \rangle$$

$$\Rightarrow \text{Area} = \frac{1}{2} |\overrightarrow{BC} \times \overrightarrow{CA}| = \frac{1}{2} \sqrt{64+4+16} = \sqrt{21}.$$

(3) Find the equation of the plane $\Pi$ containing $A, B$ and $C$.

Normal direction: $\langle 8, 2, 4 \rangle$

$\Rightarrow$ plane eqn

$$8x + 2y + 4z = 8 \cdot 0 + 2 \cdot 1 + 4 \cdot 3 = 14$$

i.e. $4x + y + 2z = 7$.

(4) Find the point of intersection of the line through $P = (2, -3, 1)$ and $Q = (1, 1, 1)$ with the plane $\Pi$ above.

$$\vec{v} = \overrightarrow{PQ} = \langle -1, 4, 0 \rangle$$

$\Rightarrow$ line eqn $\vec{r} = \langle 1, 1, 1 \rangle + t \langle -1, 4, 0 \rangle = \langle 1-t, 1+4t, 1 \rangle$.

So at intersection,

$$4(1-t) + (1+4t) + 2 \cdot 1 = 7$$

$$\Rightarrow 4 + 1 + 2 = 7.$$

So the whole line lies in the plane!

**3 (20 pts.)** Consider surface $z^3 = xyz - 4$.

(1) What is the intersection of this surface with $xy$ plane? with $xz$ plane? with $yz$ plane?

with $xy$ plane: $z=0 \Rightarrow 0 = 0-4$. no solution.
So the surface doesnot intersect $xy$ plane.

with $xz$ plane: $y=0 \Rightarrow z^3=-4 \Rightarrow z=\sqrt[3]{4}$. A line in the $xz$ plane that is parallel to $x$-axis.

with $yz$ plane: $x=0 \Rightarrow z^3=-4 \Rightarrow z=-\sqrt[3]{4}$. A line in the $yz$ plane that is parallel to $y$-axis.

(2) Find an equation of the tangent plane to this surface at the point $(2, 3, 2)$.

surface: $z^3 - xyz = 4$ is the level set of $f = z^3 - xyz$

$\nabla f = \langle -yz, -xz, 3z^2 - xy \rangle$.

@ $(2,3,2)$, $\nabla f = \langle -6, -4, 6 \rangle$

$\therefore$ Tangent plane $-6x - 4y + 6z = -12 - 12 + 12 = -12$

i.e. $3x + 2y - 3z = 6$.

(3) Use the tangent plane determined in part (2) to get an approximate solution near $z = 2$ to the equation $z^3 = (1.95)(3.05)z - 4$.

The surface defines a function $z = g(x,y)$ near the point $(2,3,2)$.
At this point, the surface has tangent plane $3x+2y-3z=6$, i.e.

$z = x + \frac{2}{3}y - 2$.

So the solution $z$ to the given eqn, which equals $g(1.95, 3.05)$, can be approximate by the $z$-coord of the point in this tangent plane with $x=1.95$, $y=3.05$. i.e.

$$z \simeq 1.95 + \frac{2}{3} \cdot 3.05 - 2 = \frac{5.95}{3}.$$

**4 (20 pts.)** Suppose $x \geq 0, y \geq 0, z \geq 0$. Find the maximal value of the function $f(x, y, z) = xyz$ subject to the constraint $x^2 + y^2 + z = 1$.

Sol: We need to solve

$$\begin{cases} f_x = yz = \lambda \cdot g_x = \lambda \cdot 2x & (1) \\ f_y = xz = \lambda \cdot g_y = \lambda \cdot 2y & (2) \\ f_z = xy = \lambda g_z = \lambda & (3) \\ x^2 + y^2 + z = 1 & (4) \end{cases}$$

Put (3) into (1) and (2), we get

$$\begin{cases} yz = 2x^2 y & (5) \\ xz = 2xy^2 & (6). \end{cases}$$

Obviously the maximum will not be obtained for $x=0$ or $y=0$. ($\Rightarrow f = 0$ !), so

$$\left.\begin{array}{l} (5) \Rightarrow z = 2x^2 \\ (6) \Rightarrow z = 2y^2 \end{array}\right\} \Rightarrow x^2 = y^2 \quad (7).$$

It follows that (4) becomes

$$x^2 + x^2 + 2x^2 = 1 \Rightarrow 4x^2 = 1 \Rightarrow x = \frac{1}{2}. \quad (\text{Note: } x \geq 0 \, !)$$

$$\therefore \; y^2 = x^2 = \frac{1}{4} \Rightarrow y = \frac{1}{2}$$

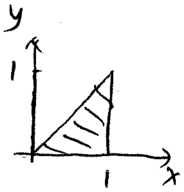
$$z = 2x^2 = 2 \cdot \frac{1}{4} = \frac{1}{2}.$$

$$\text{Maximum} = f\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) = \frac{1}{8}.$$

(Max exists since the constaint $z = 1 - x^2 - y^2$, $x, y, z \geq 0$, is a small portion of the surface $z = 1 - x^2 - y^2$, which is a closed set.)

**5 (10 pts.)** Switch the order of integration to evaluate $\int_0^1 \int_y^1 y\sqrt{1+x^3}\,dx\,dy$.

$\begin{cases} 0 \le y \le 1 \\ y \le x \le 1 \end{cases} \Rightarrow$




$\Rightarrow \begin{cases} 0 \le x \le 1 \\ 0 \le y \le x. \end{cases}$

$$I = \int_0^1 \int_0^x y\sqrt{1+x^3}\,dy\,dx$$

$$= \int_0^1 \sqrt{1+x^3} \cdot \frac{y^2}{2}\Big|_0^x \,dx$$

$$= \int_0^1 \sqrt{1+x^3} \cdot \frac{x^2}{2}\,dx$$

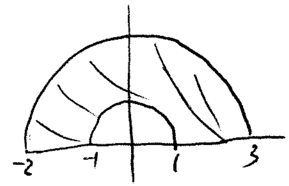
$$= \frac{1}{6}\int_0^1 \sqrt{1+u}\,du \qquad \leftarrow u = x^3.$$

$$= \frac{1}{6}\,\frac{2}{3}(1+u)^{3/2}\Big|_0^1$$

$$= \frac{1}{9}\left(\sqrt{8} - 1\right)$$

$$= \frac{1}{9}\left(2\sqrt{2} - 1\right).$$

**6 (20 pts.)** Consider the upper half of the annulus $1 \le x^2 + y^2 \le 9, y \ge 0$, with mass density $\rho(x, y) = \frac{y}{x^2+y^2}$.




$1 \le r \le 3$
$0 \le \theta \le \pi$

(1) Find the mass of this half annulus.

$$\text{Mass} = \iint_D \frac{y}{x^2+y^2}\, dx\,dy$$
$$= \int_1^3 \int_0^\pi \frac{r\sin\theta}{r^2} \cdot r \, d\theta\, dr$$
$$= 2\int_0^\pi \sin\theta \, d\theta$$
$$= 4.$$

(2) Express the $x$-coordinate of the center of mass, $\bar{x}$, as an iterated integral. (You should write explicitly the integrand and the limits of integration. Do not evaluate it.)

$$\bar{x} = \frac{1}{\text{Mass}} \iint_D \frac{x \cdot y}{x^2+y^2}\, dx\,dy = \frac{1}{4}\int_1^3 \int_0^\pi \frac{r\cos\theta \cdot r\sin\theta}{r^2}\, r\, d\theta\, dr$$
$$= \frac{1}{4}\int_1^3 \int_0^\pi r\sin\theta\cos\theta \, d\theta\, dr$$

(3) Explain why $\bar{x}$ equals zero without evaluate the integral above.

$\left.\begin{array}{l} D \text{ is symmetric w.r.t. } y\text{-axis.} \\ \text{density } \rho \text{ is even w.r.t. } x. \end{array}\right\} \Rightarrow \bar{x} = 0.$

**7 (20 pts.)** Consider the vector field $\vec{F} = \langle 3x^2 - 6y^2, -12xy + 4y \rangle$.

(1) Show that $\vec{F}$ is conservative.

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = -12y - (-12y) = 0$$

$\Rightarrow \vec{F}$ is conservative

(2) Find a potential function of $\vec{F}$.

$$f(x,y) = \int_0^x P(t,0)\,dt + \int_0^y Q(x,t)\,dt$$

$$= \int_0^x 3t^2\,dt + \int_0^y (-12xt + 4t)\,dt$$

$$= x^3 + \left(-12x\cdot\frac{y^2}{2} + 4\cdot\frac{y^2}{2}\right)$$

$$= x^3 - 6xy^2 + 2y^2 .$$

(3) Let $\vec{C}$ be the curve $x = 1 + y^2(1-y)^3, 0 \le y \le 1$. Calculate $\int_{\vec{C}} \vec{F} \cdot d\vec{s}$.

$$\int_C \vec{F}\cdot d\vec{s} = \int_{\vec{C}} \nabla f \cdot d\vec{s} = f(\text{end}) - f(\text{start}) .$$

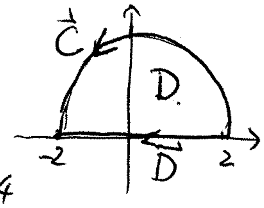
$$\left.\begin{array}{l} y=0 \Rightarrow x=1, \\ y=1 \Rightarrow x=1 \end{array}\right\} \Rightarrow \begin{array}{l} \text{start}: (1,0) \\ \text{end}: (1,1) \end{array}$$

$$\therefore \int_C \vec{F}\cdot d\vec{s} = f(1,1) - f(1,0)$$

$$= -3 - 1$$

$$= -4 .$$

**8 (20 pts.)** Let $\vec{F} = \langle x, x+y \rangle$. Let $\mathcal{C}$ be the top half of the circle $x^2 + y^2 = 4$, oriented counterclockwise. Let $\mathcal{D}$ be the line segment starting at $(2,0)$ and ending at $(-2,0)$.




(1) Evaluate $\int_{\vec{\mathcal{D}}} \vec{F} \cdot d\vec{s}$.

$\vec{D}: \vec{r} = \langle 2-t, 0 \rangle, \ 0 \le t \le 4$

$$\Rightarrow \int_{\vec{D}} \vec{F} \cdot d\vec{s} = \int_0^4 \langle 2-t, 2-t \rangle \cdot \langle -1, 0 \rangle \, dt = \int_0^4 (t-2)\, dt = 0$$

(2) Using Green's theorem and your answer to part (1) above, compute $\int_{\vec{\mathcal{C}}} \vec{F} \cdot d\vec{s}$.

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1 - 0 = 1$$

Green: $$\int_{\vec{C}-\vec{D}} \vec{F} \cdot d\vec{s} = \int_{\partial D} \vec{F} \cdot d\vec{s} = \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy = \text{Area}(D) = \frac{\pi}{2} \cdot 2^2 = 2\pi$$

$$\therefore \int_{C} \vec{F} \cdot d\vec{s} = \frac{\pi}{2} + \int_{D} \vec{F} \cdot d\vec{s} = 2\pi$$

(3) Compute $\int_{\vec{\mathcal{C}}} \vec{F} \cdot d\vec{s}$ directly, verifying your answer to part (2) above.

$\vec{C}: \vec{r} = \langle 2\cos\theta, 2\sin\theta \rangle, \ 0 \le \theta \le \pi$

$$\int_C \vec{F} \cdot d\vec{s} = \int_0^\pi \langle 2\cos\theta, 2\cos\theta + 2\sin\theta \rangle \cdot \langle -2\sin\theta, 2\cos\theta \rangle \, d\theta$$

$$= \int_0^\pi \left( -4\sin\theta\cos\theta + 4\cos^2\theta + 4\sin\theta\cos\theta \right) d\theta$$

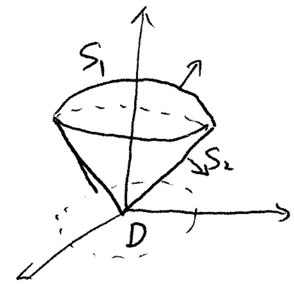
$$= \int_0^\pi 4\cos^2\theta \, d\theta$$

$$= \int_0^\pi 2(1+\cos 2\theta)\, d\theta$$

$$= 2\pi.$$

**9 (30 pts.)** Let $V$ be the solid bounded from below by $z = \sqrt{x^2+y^2}$ and bounded from above by $x^2+y^2+z^2=4$, and $S$ be its surface with outward pointing orientation.

(1) Find the volume of $V$.




$$\mathrm{Vol}(V) = \iint_D \left(\sqrt{4-x^2-y^2} - \sqrt{x^2+y^2}\right) dx\,dy$$

$$= \int_0^{\sqrt2}\int_0^{2\pi} \left(\sqrt{4-r^2} - r\right) r \, d\theta \, dr$$

$$= 2\pi\left(\int_0^{\sqrt2} \sqrt{4-r^2}\, r\, dr - \frac{2\sqrt2}{3}\right)$$

$$= \cancel{2}\pi \cdot \frac{2}{3}(8-2\sqrt2) - 2\pi\cdot\frac{2\sqrt2}{3}$$

$$= \frac{2\pi}{3}(8-4\sqrt2)$$

At intersection: $x^2+y^2+\left(\sqrt{x^2+y^2}\right)^2 = 4$

$\Rightarrow x^2+y^2=2$.

$\therefore D: x^2+y^2 \le 2$.

(2) Find the area of the surface $S$.

$S = S_1 + S_2$.

$\mathrm{Area}(S_1) = \int_0^{\frac{\pi}{4}}\int_0^{2\pi} 4\sin\phi \, d\theta\, d\phi = (8 - 4\sqrt2)\pi$.

$\mathrm{Area}(S_2) = \iint_D \sqrt{\left(\frac{x}{\sqrt{x^2+y^2}}\right)^2 + \left(\frac{y}{\sqrt{x^2+y^2}}\right)^2 + 1}\, dx\,dy = \iint_D \sqrt2\, dx\,dy = \sqrt2 \cdot \mathrm{Area}(D) = 2\sqrt2\pi$.

$\therefore \mathrm{Area}(S) = (8-2\sqrt2)\pi$.

(3) Find the flux of $\vec{F} = \langle 2x+yz, 2y-zx, z-3xy\rangle$ through the surface $\vec{S}$.

$\mathrm{div}\vec{F} = 2+2+1 = 5$

$$\mathrm{Flux} = \iint_S \vec{F}\cdot d\vec{S} = \iiint_V \mathrm{div}\vec{F}\, dx\,dy\,dz = 5\,\mathrm{Vol}(V)$$

$$= \frac{10\pi}{3}(8-4\sqrt2)$$

**10 (10 pts.) (This is only a bonus problem. Do other problems first!)**

Suppose $f = f(x)$ is a smooth function. Prove:

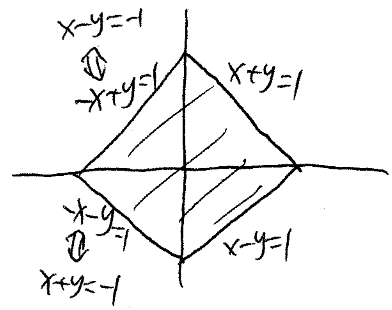
$$\iint\limits_{|x|+|y|\le 1} f(x+y)dxdy = \int_{-1}^{1} f(u)du.$$

<u>Sol</u>: $|x|+|y| \le 1$: If $x, y \ge 0$: $x+y \le 1$

If $x<0, y>0$: $-x+y \le 1$

If $x<0, y<0$: $-x-y \le 1$

If $x>0, y<0$: $x-y \le 1$




So $D$: $-1 \le x+y \le 1$, $-1 \le x-y \le 1$.

Let $u = x+y$, $v = x-y$. Then $x = \frac{u+v}{2}$, $y = \frac{u-v}{2}$.

$$J = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{4} - \frac{1}{4} = -\frac{1}{2}.$$

$$\therefore \iint\limits_D f(x+y)\,dxdy = \int_{-1}^{1}\int_{-1}^{1} f(u)\cdot\frac{1}{2}\,dvdu$$

$$= \int_{-1}^{1} f(u)\cdot\frac{1}{2}\cdot 2\,du$$

$$= \int_{-1}^{1} f(u)\,du.$$