Math 202 Practice Final Exam Solutions, Spring 2011

1a. (10pts) Find the equation of plane passing through the points $P = (2, 0, 0)$, $Q = (0, -1, 0)$, $R = (0, 0, 3)$.
Find two vectors in the plane, say $\vec{RP} =< 2, 0, -2 >$, $\vec{RQ} =< 0, -1, -3 >$ and then a normal vector to the plane is $\vec{N} = \vec{RP} \times \vec{RQ} =< -3, 6, -2 >$. hence the equation of the plane is $-3x + 6y - 2(z - 3) = 0$.

b. (10pts) Find the distance from the origin to this plane.
The vector from the origin to R is $3\hat{k}$ so the distance is

$$\text{distance} \quad = \frac{|3\hat{k} \cdot \vec{N}|}{|\vec{N}|} = \frac{6}{7} \ .$$

2. (15pts) Let C be the curve in R^3 which is the image of $\vec{c}(t) =< t^2, t^2, 2t >$, $0 \le t \le 1$. Let $f(x, y, z) = z\frac{x^4+1}{y^4+1}$. Find $\int_C f \ ds$. Simplify your answer.

$$\vec{c}'(t) =< 2t, , 2t, 2 >, \ |\vec{c}'(t)| = 2\sqrt{1 + 2t^2}, \ f(\vec{c}(t)) = 2t \ \ (\text{since } x = y = t^2) \ .$$

So

$$\int_C f \ ds = \int_0^1 2t \cdot 2\sqrt{1 + 2t^2}, \ dt = \frac{2}{3}(1 + 2t^2)^{\frac{3}{2}}|_0^1 = \frac{2}{3}(3^{\frac{3}{2}} - 1) \ .$$

3. (15pts) Let $\vec{F}(x, y, z) =< 2x - y^2 + z, x - y - z^3 >$ be a differentiable mapping from R^3 to R^2 and let $\vec{c}(t)$ ibe a curve in R^3 with $\vec{c}(0) =< 1, 1, 2 >$, $\vec{c}'(0) =< 0, -1, 3 >$. Compute $D[\vec{F} \circ \vec{c}](0)$.

$$D\vec{F}(1, 1, 2) = \begin{pmatrix} 2 & -2y & 1 \\ 1 & -1 & -3z^2 \end{pmatrix} |_{(1,1,2)} = \begin{pmatrix} 2 & -2 & 1 \\ 1 & -1 & -12 \end{pmatrix} \ .$$

$$D[\vec{F} \circ \vec{c}](0) = D\vec{F}(1, 1, 2) \begin{pmatrix} 0 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 & -2 & 1 \\ 1 & -1 & -12 \end{pmatrix} \begin{pmatrix} 0 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 5 \\ -35 \end{pmatrix} \ .$$

So $D[\vec{F} \circ \vec{c}](0) =< 5, -35 >$.

4. Let $h(x, y) = \frac{15}{x^2+2y^2+2}$ denote the height function of a mountain at the point (x,y) of the xy plane.
a. (10pts) If a climber is at the point (1,1,3), and wants to descend most rapidly, in what horizontal (unit) direction should he go?

$$h_x = \frac{-15 \cdot 2x}{(x^2 + 2y^2 + 2)^2}, \ h_y = \frac{-15 \cdot 4y}{(x^2 + 2y^2 + 2)^2}, \ \nabla h(1, 1) =< -\frac{30}{25}, -\frac{60}{25} >= -\frac{6}{5} < 1, 2 > \ .$$

The unit (horizontal) direction of fastest descent is $-\frac{\nabla h}{|\nabla h|} = \frac{<1,2>}{\sqrt{5}}$.
b. (10pts) What is the equation of the tangent plane at the climber's location?

From part a., the equation of the tangent plane is $z = 3 - \frac{6}{5}(x-1) - \frac{12}{5}(y-1)$.

5. (15pts) Let $D = \{(x,y) : x^2 + \frac{y^2}{4} = 1,\ x \geq 0, y \geq 0\}$ be the part of the interior of the ellipse in the first quadrant and let C be the boundary of D oriented counterclockwise. Evaluate directly $\int_C \vec{F} \cdot \vec{ds}$ where $\vec{F}(x,y) =< y, -x >$.

The ellipse part of the boundary can be parametrized as $x = \cos t,\ y = 2\sin t,\ 0 \leq t \leq \frac{\pi}{2}$ and the line integral over the straight parts of the boundary vanishes so

$$\int_C y dx - x dy = \int_0^{\frac{\pi}{2}} (-2\sin^2 t - 2\cos^2 t)\ dt = -\pi\ .$$

6. (20pts) Let $f(x,y) = 3x^2 + \frac{3}{2}y^2 + yx^2$. Find all critical points of f(x,y) and classify them as local max, local min or saddle point. Justify .

$$f_x = 6x + 2xy = 2x(3+y) = 0,\ f_y = 3y + x^2 = 0,$$

so the critical points of $f(x,y)$ are $(0,0)$, $(3,-3)$, $(-3,-3)$.
The Hessian $D^2 f(x,y) = \begin{pmatrix} 6+2y & 2x \\ 2x & 3 \end{pmatrix}$. Using the second derivative test, we see that (0,0) is a relative min and the other critical points are saddles.

7. (15pts) Let $\vec{F} =< x+yz, y+xz, z+xy >$ and let $\vec{c}(t) =< \cos \pi t, 2\sin \pi t, (1+t)^2 >$, $0 \leq t \leq 1$. Calculate the line integral $\int_C \vec{F} \cdot \vec{ds}$.

$\vec{F}$ is conservative with potential function $\phi = \frac{x^2}{2} + xyz + \frac{y^2}{2} + \frac{z^2}{2}$.
Since $\vec{c}(1) =< -1, 0, 4 >$ and $\vec{c}(0) =< 1, 0, 1 >$,

$$\int_C \vec{F} \cdot \vec{ds} = \phi(-1,0,4) - \phi(1,0,1) = \frac{17}{2} - 1 = \frac{15}{2}\ .$$

8. (15pts) Let D be the y-simple domain in the xy plane defined by

$$D = \{(x,y) : 0 \leq x \leq 2\pi,\ 0 \leq y \leq 2 + \cos x\}\ .$$

Let $\vec{F}(x,y) =< xe^x - y^2, \sin y >$. Evaluate $\int_C \vec{F} \cdot d\vec{s}$ using Green's theorem, where C is the boundary of D oriented counterclockwise.

$$\int_C \vec{F} \cdot \vec{ds} = \int\int_D (Q_x - P_y)\ dA = \int\int_D 2y\ dA = \int_0^{2\pi}\int_0^{2+\cos x} 2y\ dydx$$

$$= \int_0^{2\pi} (2+\cos x)^2\ dx = \int_0^{2\pi} (4 + 4\cos x + \frac{1+\cos 2x}{2})dx = (4 + \frac{1}{2}) \cdot 2\pi = 9\pi\ .$$

9. (15pts) Let W be the solid region consisting of the part of the unit ball $x^2+y^2+z^2 \leq 1$ in the first octant $(x \geq 0,\ y \geq 0,\ z \geq 0)$. Let S be the boundary of W oriented by the outward unit normal. Calculate

$$\int\int_S \vec{F} \cdot d\vec{S} \text{ where } \vec{F}(x,y,z) =< -xyz, y^2 z + x, e^x >\ .$$

Hint: Use the divergence theorem to convert this to a triple integral and then use spherical coordinates.

$$\int\int_S \vec{F}\cdot d\vec{S} = \int\int\int_W \text{div } \vec{F}\ dV = \int\int\int_W (-yz+2yz)\ dV = \int\int\int_W yz\ dV$$
$$= \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^1 (\rho\sin\phi\sin\theta\ \rho\cos\phi)\rho^2\sin\phi\ d\rho d\theta d\phi$$
$$= \frac{1}{5}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \sin^2\phi\cos\phi\sin\theta\ d\theta d\phi = \frac{1}{5}\cdot 1\frac{\sin^3\phi}{3}|_0^{\frac{\pi}{2}} = \frac{1}{15}\ .$$

10. (15 pts) Find the area of the part of the cylinder $x^2+y^2=1$ that lies above the plane z=0 and below the surface $z=4+x^2-y^2$. Hint: Recall $\cos^2\theta = \frac{1+\cos 2\theta}{2}$, $\sin^2\theta = \frac{1-\cos 2\theta}{2}$.

The surface S is parametrized by

$\vec{X}(\theta,z) =< \cos\theta, \sin\theta, z >$ in $D = \{(\theta,z) : 0\le z\le 4+\cos^2\theta-\sin^2\theta,\ 0\le\theta\le 2\pi\ .\}$.

(Recall $\vec{X}_\theta\times\vec{X}_z =< \cos\theta,\sin\theta,0 >$, $|\vec{X}_\theta\times\vec{X}_z| = 1$.)

Hence

$$A = \int\int_D d\theta dz = \int_0^{2\pi}\int_0^{4+\cos^2\theta-\sin^2\theta} d\theta dz = \int_0^{2\pi}(4+\cos^2\theta-\sin^2\theta)\ d\theta$$
$$= 8\pi + \int_0^{2\pi}\cos 2\theta\ d\theta = 8\pi\ .$$

11. (15pts) Find the area of the graph of the function $f(x,y) = \frac{2}{3}(x^{\frac{3}{2}}+y^{\frac{3}{2}})$ over the unit square $0\le x\le 1$, $0\le y\le 1$.

$$A = \int_0^1\int_0^1 \sqrt{1+f_x^2+f_y^2}\ dxdy = \int_0^1\int_0^1\sqrt{1+x+y}\ dxdy$$
$$\int_0^1 \frac{2}{3}(1+x+y)^{\frac{3}{2}}|_0^1\ dy = \int_0^1\frac{2}{3}[(2+y)^{\frac{3}{2}}-(1+y)^{\frac{3}{2}}]\ dy$$
$$= \frac{2}{3}\cdot\frac{2}{5}[(2+y)^{\frac{5}{2}}-(1+y)^{\frac{5}{2}}]|_0^1 = \frac{4}{15}(3^{\frac{5}{2}}-2\cdot 2^{\frac{5}{2}}+1)\ .$$

12. (20 pts) Let $\vec{F} =< x^2+y-4, 3xy, 2xz+z^2$ and let S be the hemisphere $x^2+y^2+z^2=16$, $z\ge 0$ with S oriented by the upward normal. Use Stokes' theorem to calculate $\int\int_S \nabla\times\vec{F}\cdot d\vec{S}$.

$$\int\int_S\nabla\times\vec{F}\cdot d\vec{S} = \int_C \vec{F}\cdot d\vec{s} = \int_C (x^2+y-4)dx + 3xy\ dy$$
$$= \int_C ydx+3xy\ dy = \int_0^{2\pi}(-16\sin^2\theta + 3\cdot 64\cos^2\theta\sin\theta)\ d\theta = -16\pi\ .$$

Note that we have taken a shortcut: $\int_C (x^2-4)\ dx = 0$.