Homework 5. Cong. Ma.

3.4 Constrained Extrema & Lagrange Multiplier.

24 $f(x,y,z) = x + yz$

$\nabla f = (1, z, y) \neq (0,0,0)$ for any point.

let $g(x,y,z) = x^2+y^2+z^2$ $\nabla g = (2x, 2y, 2z)$

$\begin{cases} (1, z, y) = \lambda(2x, 2y, 2z) \\ x^2+y^2+z^2 = 1 \end{cases}$

$\Rightarrow 4\lambda^2 = 1 \Rightarrow \lambda = \pm\frac{1}{2}$ then $x = \pm 1$

and we can find $y = z = 0$ by using $x^2+y^2+z^2=1$

$\Rightarrow$ we only have $(1,0,0)$ & $(-1,0,0)$ as our candidates.

$f(1,0,0) = 1$ $f(-1,0,0) = -1$

then Max at $(1,0,0)$ Min at $(-1,0,0)$.

38 $Q(x,y) = xy - x - y + 1$

$g(x,y) = px + qy \leq B$

$\nabla Q = (y-1, x-1) = \lambda \nabla g = \lambda(p, q)$

$\begin{cases} y-1 = \lambda p \quad x-1 = \lambda q \\ px + qy = B \end{cases}$

then $\lambda = \dfrac{B-p-q}{2pq}$ $\Rightarrow y = \dfrac{B-p+q}{2q}$ $x = \dfrac{B-q+p}{2p}$

therefore $\dfrac{y}{x} = \dfrac{p}{q} \cdot \dfrac{B-p+q}{B-q+p}$ is the ratio for max production.

4.1 12 $c'(0) = (1, 1, -2)$

Acceleration and Newton's Second Law.

$c''(t) = (0, 0, 6) \Rightarrow c'(t) = (A, B, 6t + C)$

$\Rightarrow A = 1$ $B = 1$ $C = -2$

$\Rightarrow c'(t) = (1, 1, 6t-2) = v(t)$

$c(t) = (t + A, t + B, 3t^2 - 2t + c)$

$c(0) = (3, 4, 0)$

$\Rightarrow c(t) = (t+3, t+4, 3t^2 - 2t)$

20. If Min or Max of $\|r(t)\|$ happens.

then so is then $\|r(t)\|^2$ for its max, min.

then $(\|r(t)\|^2)' = 2r'(t) \cdot r(t) = 0$.

$\Rightarrow$ $r'(t) \cdot r(t) = 0 \Rightarrow$ they are perpendicular to each other.

26 $F: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ C^1 function.

$$(*) = \frac{d}{dt}[m c(t) \times v(t)] = m \frac{d}{dt}[c(t) \times v(t)]$$

$$= m[\frac{dc(t)}{dt} \times v(t) + c(t) \times \frac{dv(t)}{dt}]$$

$$\frac{dc(t)}{dt} = v(t) \quad \frac{dv(t)}{dt} = a(t)$$

$$v(t) \times v(t) = 0$$

$$(*) = m \cdot c(t) \times a(t)$$

$$= c(t) \times m a(t) \Rightarrow (*) = c(t) \times F(c(t))$$

4.2 Arc Length.

4. $$L(c) = \int_1^2 \|c'(t)\| dt = \int_1^2 \sqrt{1^2 + (\sqrt{2}t^{\frac{1}{2}})^2 + t^2}\, dt$$

$$= \int_1^2 \sqrt{(t+1)^2}\, dt$$

$$= \frac{t^2}{2} + t \Big|_1^2 = \frac{5}{2}.$$

8. $$\int_0^{2\pi R} \|c'(t)\| dt = \int_0^{2\pi R} R\sqrt{2-2\cos t}\, dt$$

$$= R\int_0^{2\pi R} 2\sqrt{\frac{1-\cos t}{2}}\, dt$$

$$= 4R\int_0^{2\pi R} \sin\frac{t}{2}\, d\frac{t}{2}$$

$$= 4R(-\cos\frac{t}{2})\Big|_0^{2\pi R}$$

$$= 8R = 4 \times \text{diameter}.$$

17(d) for $c(t) = \frac{1}{\sqrt{2}}(\cos t, \sin t, t)$

$c'(t) = \frac{1}{\sqrt{2}}(-\sin t, \cos t, 1)$ $c''(t) = \frac{1}{\sqrt{2}}(-\cos t, -\sin t, 0)$

by (b) $\|c''(t)\| = \sqrt{\frac{\cos^2 t}{2} + \frac{\sin^2 t}{2}} = \frac{1}{\sqrt{2}}$. is the curvature.

18 for $\begin{cases} l'(t) = \vec{v} \\ l''(t) = 0 \end{cases}$

Using 17 (a) (b) $\Rightarrow$ curvature is zero

by $l''(t) = 0$ Alternative is use formular in 17(c). it works as well.

22 torsion is defined by $\frac{dB}{ds} = -\tau N$.

Here if the path is on a plane. then use formular to compute T. and N found out that B is constant normal to the plane

$\Rightarrow \frac{dB}{ds} = 0 = -\tau N$.

But $N \neq 0 \Rightarrow \tau = 0$.

torsion is zero.

24 for $A = (0,0,0,0)$ $B = (x, 0, 0, t_1)$ $C = (0, 0, 0, t_2)$

$T_{AB} = \frac{1}{c}\int_0^1 \sqrt{x^2 + c^2 t_1^2}\,dx = \frac{1}{c}\sqrt{-x^2 + c^2 t_1^2}$

$T_{BC} = \frac{1}{c} \int_0^1 \sqrt{-x^2 + c^2(t_2 - t_1)^2}$ $T_{AC} = \frac{1}{c}\sqrt{c^2 t_2^2} = \frac{1}{c}(c t_2)$

~~$T_{AB} + T_{BC} < \frac{1}{c}\sqrt{-x^2 + c^2 t_2}$~~

$P_{AB} + P_{BC} < P_{AC}$

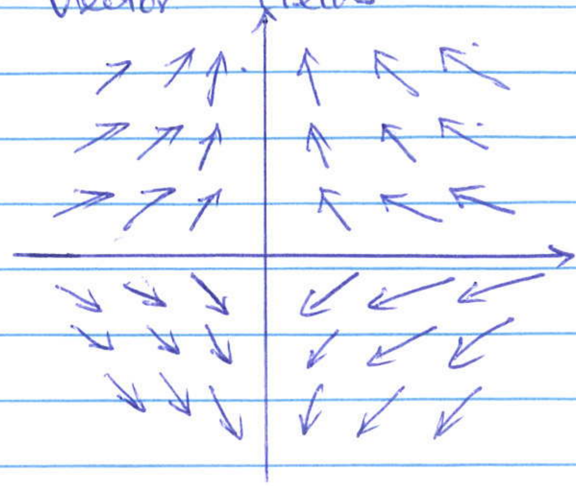
$\Leftrightarrow \sqrt{-x^2 + c^2 t_1^2} + \sqrt{-x^2 + c^2(t_2 - t_1)^2} < c t_2$

$\Leftrightarrow -x^2 + c^2 t_1^2 < c^2 t_1^2$

$\Leftrightarrow -x_1^2 < 0$ ∎

4.3 Vector Fields

4.



10. Both of them are not defined at $(0,0)$.

for (a) if we compute it at $(1,1)$

$\rightarrow \langle \frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}} \rangle$ then it's (i)

(b) if we compute it at $(1,1)$

$\rightarrow \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$ then it's (ii)

18 $c(t) = (\frac{1}{t^3}, e^t, \frac{1}{t})$; $F(x,y,z) = (-3z^4, y, -z^2)$

$c'(t) = (\frac{-3}{t^4}, e^t, \frac{-1}{t^2})$

$F(c(t)) = (\frac{-3}{t^4}, e^t, \frac{-1}{t^2})$

$\Rightarrow c'(t) = F(c(t))$