Series tests

Definition. Let $\{a_n\}$ be a sequence of numbers. The n-th *partial sum* of this sequence is

$$s_n := \sum_{i=1}^{n} a_i.$$

We write $\sum a_n$ to denote the sequence of partial sums, $\{s_n\}$, and say $\sum a_n = s$ if $\lim s_n = s$. Thus, $\sum a_n$ converges iff its sequence of partial sums converges. A series *diverges* if it does not converge.

Examples. The series $\sum_{n=1} \frac{1}{n(n+1)} = 1$. To see this we consider the sequence of partial sums. First note that

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}.$$

Thus, the n-th partial sum for the series is

$$s_n = \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n}\right) + \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$$= \frac{1}{1} - \frac{1}{n+1}.$$

The above sum is called a *telescoping sum*—all its intermediate terms cancel. We have

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} := \lim_{n\to\infty} s_n = \lim_{n\to\infty} \left(\frac{1}{1} - \frac{1}{n+1}\right) = 1.$$

The sequence of partial sums of the series $\sum_{n=1}^{\infty} (-1)^n$ is

$$\begin{aligned}
s_1 &= (-1)^1 = -1 \\
s_2 &= (-1)^1 + (-1)^2 = -1 + 1 = 0 \\
s_3 &= (-1)^1 + (-1)^2 + (-1)^3 = -1 + 1 - 1 = -1 \\
s_4 &= (-1)^1 + (-1)^2 + (-1)^3 + (-1)^4 = -1 + 1 - 1 + 1 = 0 \\
&\vdots
\end{aligned}$$

In other words, the sequence of partial sums for $\sum_{n=1}^{\infty} (-1)^n$ is $-1, 0, -1, 0, -1, 0, -1, \ldots$. This sequence does not converge since it has constant subsequences $\{-1\}$ and $\{0\}$ converging to different values. Therefore, $\sum_{n=1}^{\infty} (-1)^n$ diverges.

1. If $\sum a_n$ converges, then $\lim a_n = 0$. (The converse does not hold: consider the harmonic series, $\sum_{n=1}^{\infty} \frac{1}{n}$.)

   Example. The series $\sum_{n=1}^{\infty}(-1)^n$ diverges since $\lim_{n\to\infty}(-1)^n$ doesn't exist. In particular, the limit of these terms of the series is not 0.

   The series $\sum_{n=1}^{\infty} \cos(2/n)$ does not converge since

   $$\lim_{n\to\infty} \cos\left(\frac{2}{n}\right) = \cos(0) = 1 \neq 0.$$

   Warning: This test cannot be used to prove convergence. For instance, the harmonic seres $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges even though $\lim_{n\to\infty} \frac{1}{n} = 0$.

2. **Comparison test.** Let $\sum a_n$ and $\sum b_n$ be series of nonnegative terms with $a_n \leq b_n$ for all n. Then if $\sum b_n$ converges, so does $\sum a_n$. (It follows that if $\sum a_n$ diverges, so does $\sum b_n$.

   Example. We saw above that

   $$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1.$$

   Then, since

   $$0 \leq \frac{1}{(n+1)^2} \leq \frac{1}{n(n+1)}$$

   it follows that $\sum_{n=1}^{\infty} \frac{1}{(n+1)^2}$ converges (and, hence, $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges).

   On the other hand, we can use the comparison test to show that $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges. That's because

   $$\frac{1}{n} \leq \frac{1}{\sqrt{n}},$$

   and $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (as we will see in class).

3. **Limit comparison test.** Let $\sum a_n$ and $\sum b_n$ be series of positive terms, and suppose that

   $$\lim \frac{a_n}{b_n} = L \neq 0.$$

   Then $\sum a_n$ converges iff $\sum b_n$ converges. (Intuitively, since $L \neq 0$, the terms a_n and b_n die off at the same rate, and thus their series both converge or they both diverge.)

Examples. Consider

$$\sum_{n=1}^{\infty} \frac{6n+7}{2n^2+4}.$$

For n large, we have

$$\frac{6n+7}{2n^2+4} \approx \frac{6n}{2n^2} = \frac{3}{n}.$$

Since $\sum_{n\to\infty} \frac{1}{n}$ diverges, so does $\lim_{n\to\infty} \frac{3}{n}$. We would expect that $\sum_{n=1}^{\infty} \frac{6n+7}{2n^2+4}$ also diverges. The limit comparison theorem makes this precise. Let $a_n = \frac{6n+7}{2n^2+4}$ and $b_n = \frac{1}{n}$. Then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \left(\frac{6n+7}{2n^2+4} \cdot \frac{n}{1} \right) = \lim_{n\to\infty} \frac{6n^2+7n}{2n^2-4} = 3 \neq 0.$$

Hence, $\sum_{n=1}^{\infty} \frac{6n+7}{2n^2+4}$ diverges by limit comparison with the harmonic series.

Let's try to use limit comparison to compare the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ and $\sum_{n\to\infty} \frac{1}{n}$:

$$\lim_{n\to\infty} \left(\frac{1}{n^2} \cdot \frac{n}{1} \right) = \lim_{n\to\infty} \frac{1}{n} = 0.$$

Since the limit is 0, the test is inconclusive.

4. **Absolute convergence.** Let $\sum a_n$ be a complex sequence. If $\sum |a_n|$ converges, so does $\sum a_n$, and we

   NOTE: The converse does not hold. For example, consider the alternating harmonic series, $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$. It converges by the alternating series test (below), but taking the absolute value of the terms results in the harmonic series, $\sum_{n=1}^{\infty} \frac{1}{n}$, which diverges. If $\{a_n\}$ is a sequence of positive terms and $\sum (-1)^n a_n$ converges but not absolutely, we say $\sum a_n$ *converges conditionally.* Thus, the alternating harmonic series is conditionally convergent.

   Examples. Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, both of the following series are absolutely convergent (and hence convergent):

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \qquad \sum_{n=1}^{\infty} \frac{\cos(n) + i\sin(n)}{n^2}.$$

   (Note that $|\cos(n) + i\sin(n)| = 1$.

5. **Ratio test.** Let $\sum a_n$ be a series of positive terms and suppose

$$\lim \frac{a_{n+1}}{a_n} = R.$$

If $R < 1$, the series converges. If $R > 1$ or $R = \infty$, the series diverges. If $R = 1$, the test is inconclusive.

Examples.

(a) The series

$$\sum_{n=0}^{\infty} \frac{1}{n!} = 1 + \frac{1}{2} + \frac{1}{3!} + \frac{1}{4!} + \dots$$

is convergent by the ratio test:

$$\lim_{n\to\infty} \left(\frac{1}{(n+1)!} \Big/ \frac{1}{n!} \right) = \lim_{n\to\infty} \frac{n!}{(n+1)!} = \lim_{n\to\infty} \frac{1}{n+1} = 0 < 1.$$

(b) The series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges and the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges. The ratio test applied to either results in a limit of $R = 1$. This example illustrated that when $R = 1$, the ratio test is inconclusive.

(c) The ratio test does not immediately apply to $\sum_{n=1}^{\infty} (-1)^n n (\frac{1}{2})^n$ since the terms of the series are not all positive. Nevertheless, the ratio test is relevant: taking the absolute value of each term yields a series with positive terms to which we can apply the ratio test:

$$\lim_{n\to\infty} \left((n+1) \left(\frac{1}{2}\right)^{n+1} \Big/ n \left(\frac{1}{2}\right)^{n} \right) = \lim_{n\to\infty} \frac{n+1}{2n} = \frac{1}{2} < 1.$$

Hence, the original is not only convergent, it is absolutely convergent. (Recall, though, that if the ratio test had indicated the series of positive terms diverged, the original series could still be convergent. A series can converge even though it is not absolutely convergent.)

6. **Root test.** Let $\sum a_n$ be a series of nonnegative terms. Suppose that $\lim a_n^{1/n} = \alpha$. If $\alpha < 1$, the series converges. If $\alpha > 1$, the series diverges. If $\alpha = 1$, the test is inconclusive.

We will not use this test much.

7. **Integral test.** Suppose $f(x)$ is a continuous, positive, decreasing function whose domain contains $[1, \infty)$. Then $\sum f(n)$ converges iff $\int_1^\infty f(x)\,dx := \lim_n(\int_1^n f(x)\,dx)$ converges.

   Examples.

   (a) The function $f(x) = \frac{1}{x}$ is continous, positive, and decreasing on $(0, \infty)$. We have

$$\int_1^\infty \frac{1}{x}\,dx = \lim_n \int_1^n \frac{1}{x}\,dx = \lim_n(\ln(n) - \ln(1)) = \lim_n(\ln(n)) = \infty.$$

   Since the integral diverges, so does

$$\sum_{n=1}^{\infty} f(n) = \sum_{n=1}^{n} \frac{1}{n}.$$

   So the integral test gives another proof that the harmonic series diverges.

   (b) Here we apply the integral test to prove that $\sum_{n=1}^\infty \frac{1}{n^2}$ converges:

$$\begin{aligned}\int_1^\infty \frac{1}{x^2}\,dx &= \lim_n \int_1^n \frac{1}{x^2}\,dx \\ &= \lim_n \left(-\frac{1}{x}\right)\Bigg|_{x=1}^{n} \\ &= \lim_n \left(-\frac{1}{n} + \frac{1}{1}\right) \\ &= 1.\end{aligned}$$

8. ***p*-series test.** Let $p \in \mathbb{R}$. The series

$$\sum \frac{1}{n^p}$$

   converges iff $p > 1$. (This result follows easily from the integral series test.)

   Examples. The following series converge by the p-series test:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \qquad \sum_{n=1}^{\infty} \frac{1}{n^{1.03}}.$$

The following series diverge by the p-series test:

$$\sum_{n=1}^{\infty} \frac{1}{n} \qquad \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}.$$

The p-series test is agnostic on the convergence of

$$\sum_{n=1}^{\infty} \frac{1}{n^{1+1/n}}.$$

9. **Alternating series test.** If (a_n) be a decreasing sequence of nonnegative numbers, and $\lim a_n = 0$, then

$$\sum (-1)^{n+1} a_n$$

converges.

Examples.

(a) The alternating harmonic series,

$$\sum (-1)^{n+1} \frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$$

converges by the alternating series test since the sequence $\{1/n\}$ is a decreasing sequence of nonnegative numbers with limit 0.

(b) The alternating series test does not apply to

$$\sum_{n=1}^{\infty} (-1)^n \frac{n}{3n+1}$$

since $\lim_n \frac{n}{3n+1} = \frac{1}{3} \neq 0$.

(c) There are examples of divergent alternating series for which $a_n \to 0$ however the a_n are not decreasing.

(d) Consider

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sqrt{n}}{3n+4}.$$

The sequence $\left\{ \frac{\sqrt{n}}{3n+4} \right\}$ increase for its first few terms but then decreases after that, and the limit of the sequence if 0. So the alternating series test, does apply, and we conclude that the series converges.

10. **Geometric series.** Let $r \in \mathbb{C}$. The series $\sum_{n=0}^{\infty} r^n$ converges if and only if $|r| < 1$. When $|r| < 1$,

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}.$$

More generally, for $a \in \mathbb{C}$ and $k \in \mathbb{N}$, if $|r| < 1$, then

$$\sum_{n=k}^{\infty} ar^n = \frac{ar^k}{1-r}.$$

Examples.

(a)

$$\sum_{n=0}^{\infty} \left(\frac{i}{2}\right)^n = \frac{1}{1-\frac{i}{2}} = \frac{2}{2-i} = \frac{2}{2-i} \cdot \frac{2+i}{2+i} = \frac{4}{5} + \frac{1}{5}i.$$

(b)

$$\sum_{n=2}^{\infty} 5\left(\frac{2}{3}\right)^n = 5\left(\frac{2}{3}\right)^2 \frac{1}{1-\frac{2}{3}} = \frac{20}{3}.$$

(c) $\sum_{n=1}(4i)^n$ diverges since $|4i| = 4 \geq 1$.