MATH 112: A guide to mathematical writing

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Mathematical thinking is based on the operations of logic. We list some of them below, with common symbolic representations. While these symbols can be useful in shorthand, boardwork and scratchwork, it is generally best to write your mathematical reasoning using words and complete sentences.

$\forall$: for all $\exists$: there exists $\neg$: not

$\Longrightarrow$: implies, or "if ..., then ..."   ST : "such that"

$\Longleftrightarrow$: if and only if (sometimes abbreviated "iff")

I recommend that you avoid the symbols $\wedge$ and $\vee$ for "and" and "or", since it is clearer and just as easy to write out the words.

In this course, we will write mathematical proofs to justify true mathematical statements. A statement is a declarative sentence with a well-defined meaning that is either true or false. For example,

"12 is a prime number."

is a mathematical statement, but

"12 is a groovy number."

is not a mathematical statement (unless we have given a precise definition of what it means for a number to be groovy). Mathematical proofs are written using complete sentences in a human language (for this course, English please!). A proof starts from known assumptions, then performs logical arguments to arrive at a goal. It is a good idea to remind the reader of where you are starting, where you are going, and where you shall arrive. Remember the old dictum on oration:

"Tell them what you will say, then say it, then tell them what you have said."

Mathematics can be technical, cumbersome, and confusing, so it is always a good idea to be as clear as possible in your explanations of the logical steps in a proof.

One useful method of proof is proof by contradiction. If you want to prove that a proposition Q is true under the hypothesis (or a list of hypotheses) P, it is often easier to assume to the contrary that Q is false and then reach a contradiction with the hypothesis P. Since we do not allow contradictions in our logical world, we have no choice but to conclude that the assumption that Q was false was incorrect, thus proving Q. In symbols, the implication

$$P \Longrightarrow Q$$

is equivalent to its contrapositive

$$\neg Q \implies \neg P.$$

If you have written a proof by contradiction, it is a good idea to check afterwards whether a simpler proof that proceeds directly from P to Q is possible.

Exercise. Find examples of propositions P and Q that show that the implication

$$P \implies Q$$

is not always equivalent to its converse

$$Q \implies P.$$

Let's consider some examples of proofs.

Proposition. *The square root of* 2 *is an irrational number.*

Proof. To prove that $\sqrt{2}$ is irrational, we must show that it is not a rational number. We will proceed by using the method of proof by contradiction: we will assume that $\sqrt{2}$ is a rational number, then then derive a contradiction, thereby showing that our assumption must be false.

Assume that $\sqrt{2}$ is rational. This means that there exists integers m and n such that $n \neq 0$ and $\sqrt{2} = m/n$. By dividing out by the common divisors of m and n, we may assume that m and n have no common divisors greater than 1 (in other words, the fraction m/n is in reduced form). This means that if d is an integer larger than 1, then d cannot divide both m and n.

Squaring both sides of the equation $\sqrt{2} = m/n$, we see that:

$$2 = (\sqrt{2})^2 = \frac{m^2}{n^2}.$$

Rearranging this expression, we arrive at:

$$2n^2 = m^2.$$

Notice that since 2 divides the left hand side of the equation, 2 must also divide the right hand side of the equation. Recall that 2 is a prime number, i.e. it has no divisors besides 1 and itself. This means that if 2 divides a product of integers ab, then there is no way to split up the divisors of 2 among the two factors a and b, and so 2 must divide either a or b. In our situation, we know that 2 divides $m^2 = m \cdot m$, and so we may conclude that 2 divides m. Now we may write $m = 2m'$ for some integer m'. Substituting this into the equation above, we get:

$$2n^2 = m^2 = (2m')^2 = 4(m')^2.$$

Dividing by 2, we see that:

$$n^2 = 2(m')^2.$$

Therefore 2 divides n^2, and so because 2 is prime, 2 must divide n as well. We have shown that 2 divides both m and n, which is a contradiction to our assumption that m and n have no common divisors. From this contradiction, we conclude that our original assumption that $\sqrt{2}$ is a rational number must be false, and so $\sqrt{2}$ is irrational. □

Here is an example of a *bad* proof of a true fact.

Proposition. *The equation* $\sin^2 x(1 + \cot^2 x) = 1$ *holds for every number* x.

Proof.

$$\begin{aligned} \sin^2 x(1 + \cot^2 x) &\stackrel{?}{=} 1 \\ \sin^2 x + \sin^2 x \cdot \frac{\cos^2 x}{\sin^2 x} &\stackrel{?}{=} 1 \\ \sin^2 x + \cos^2 x &= 1, \end{aligned}$$

which is true. □

Here are some reasons why this proof is bad:

- The proof does not use complete sentences to conduct a logical argument.
- The proof starts with symbols, without saying what the objects are or why we are considering them.
- The first equation is not yet known to be true–in fact it is exactly what we want to prove. Even with the question-mark symbol over the equals sign, it is *never* a good idea to write down something that is not known to be true, unless you precede it with a qualification, such as: "We want to prove that ..."
- At the end of the proof, we arrive at something asserted to be true, but it is not clear why that proves the desired statement.

In general, I strongly discourage the use of the "horseshoe" proof method. Here is a good proof of the same fact:

Proof. We want to prove that the equation $\sin^2 x(1 + \cot^2 x) = 1$ holds for every value of x. To this end, we make some algebraic manipulations:

$$\begin{aligned} \sin^2 x(1 + \cot^2 x) &= \sin^2 x + \sin^2 x \cdot \frac{\cos^2 x}{\sin^2 x} \\ &= \sin^2 x + \cos^2 x. \end{aligned}$$

By substituting in the identity $\sin^2 x + \cos^2 x = 1$, we arrive at the desired equation

$$\sin^2 x(1 + \cot^2 x) = 1.$$

□

You might still have some complaints about the proof. For example, we did not justify why the identity $\sin^2 x + \cos^2 x = 1$ is true. It is valid to use previously known results in your proofs, but you must judge based on the context how much prior knowledge you expect the reader of your proof to have. For this course, imagine that you are writing your proofs in order to communicate your ideas to a fellow classmate. If you have already proved the identity $\sin^2 x + \cos^2 x = 1$ in the course, or in a previous one, then it is reasonable to use it in a proof.

Here is an example of a proof by induction.

Proposition. *Let n be a positive integer. Then:*

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}.$$

Proof. We will prove the identity by induction on the number n. When $n = 1$, the identity is obvious:

$$1 = \frac{1(1+1)}{2}.$$

Thus we have established the base case. Inductively assume that the identity holds for some $n \geq 1$:

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}.$$

In order to prove that the identity holds for $n + 1$, we add $n + 1$ to both sides of the above equation, then perform some algebraic manipulations:

$$\begin{aligned} 1 + 2 + 3 + \cdots + n + (n+1) &= \frac{n(n+1)}{2} + (n+1) \\ &= \frac{n(n+1) + 2(n+1)}{2} \\ &= \frac{n^2 + 3n + 2}{2} \\ &= \frac{(n+1)(n+2)}{2}. \end{aligned}$$

The combined equation proves the identity for $n + 1$, which completes the proof of the inductive step. □