MATH 112: $\mathbf{R}$ is the unique complete ordered field

John Lind • February 12, 2018

Definition. A *field* is a set F equipped with two binary operations, $+\colon F \times F \longrightarrow F$ (addition) and $\cdot\colon F \times F \longrightarrow F$ (multiplication), and two special elements $0 \in F$ and $1 \in F$ satisfying the following axioms:

(F1) 0 is the additive identity: $a + 0 = a = 0 + a$ for all $a \in F$

(F2) Addition is associative: $a + (b + c) = (a + b) + c$ for all $a, b, c \in F$

(F3) Addition is commutative: $a + b = b + a$ for all $a, b \in F$

(F4) 1 is the multiplicative identity: $a \cdot 1 = a = 1 \cdot a$ for all $a \in F$

(F5) Multiplication is associative: $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ for all $a, b, c \in F$

(F6) Multiplication is commutative: $a \cdot b = b \cdot a$ for all $a, b \in F$

(F7) Distributivity: $a \cdot (b + c) = a \cdot b + a \cdot c$ for all $a, b, c \in F$

(F8) Additive inverses exist: for every $a \in F$, there exists $b \in F$ such that $a + b = b + a = 0$

(F9) Multiplicative inverses exist for nonzero elements: if $a \in F$ and $a \neq 0$, then there exists $b \in F$ such that $a \cdot b = b \cdot a = 1$

(F10) $0 \neq 1$

We usually write $-a$ for the additive inverse of a. The subtraction operation is then defined by $c - a := c + (-a)$. Similarly, we usually write a^{-1} for the multiplicative inverse of $a \neq 0$. Division is then defined by $c/a := c \cdot (a^{-1})$. We allow ourselves to make the abbreviation $ab := a \cdot b$ for multiplication when it is convenient.

Definition. An *ordered field* is a field F equipped with a relation $<$ that satisfies the following axioms:

(O1) Trichotomy: Given elements $a, b \in F$, exactly one of the following statements is true:

$$a < b, \quad \text{or} \quad b < a, \quad \text{or} \quad a = b.$$

(O2) Transitivity: for all $a, b, c \in F$,

$$\text{if } a < b \text{ and } b < c, \quad \text{then } a < c.$$

(O3) $<$ respects addition: for all $a, b, c \in F$,

$$\text{if } a < b, \quad \text{then } a + c < b + c.$$

(O4) $<$ respects multiplication: for all $a, b, c \in F$,

$$\text{if } a < b \text{ and } 0 < c, \quad \text{then } a \cdot c < b \cdot c.$$

We define $a > b$ to mean $b < a$. We also define $a \leq b$ to mean that either $a < b$ or $a = b$, and similarly we define $a \geq b$ to mean that either $a > b$ or $a = b$.

Definition. Let F be an ordered field, and let $T \subseteq F$ be a subset of F.

- An element $M \in F$ is called an *upper bound* of T if $t \leq M$ for all $t \in T$.
- An element $m \in F$ is called a *lower bound* of T if $m \leq t$ for all $t \in T$.
- If an upper bound of T exists, then we say that T is *bounded above.* If a lower bound of T exists, then we say that T is *bounded below.* If T is bounded above and bounded below, then we say that T *is bounded.*
- An element $M \in F$ is called a *least upper bound*, or *supremum*, of T if M is an upper bound of T and it is less then or equal to every other upper bound. In other words,

  (1) $t \leq M$ for all $t \in T$, and
  (2) if $M' \in F$ satisfies $t \leq M'$ for all $t \in T$, then $M \leq M'$.

  We usually write $\sup T$ as shorthand for a least upper bound M of T.
- An element $m \in F$ is called a *greatest lower bound*, or *infimum*, of T if m is a lower bound of T and it is greater then or equal to every other lower bound. In other words,

  (1) $m \leq t$ for all $t \in T$, and
  (2) if $m' \in F$ satisfies $m' \leq t$ for all $t \in T$, then $m' \leq m$.

  We usually write $\inf T$ as shorthand for a greatest lower bound m of T.

It is important to note that a least upper bound $\sup T$ of T need not be an element of T. Similarly, a greatest lower bound $\inf T$ need not be an element of T.

It is not always the case that $\sup T \in T$ **or that** $\inf T \in T$**!**

When $\sup T \in T$, then we call $\sup T$ the *maximum* of T. When $\inf T \in T$, then we call $\inf T$ the *minimum* of T.

Definition. An ordered field F is *complete* if every nonempty subset T of F that is bounded above has a least upper bound $\sup T \in F$.

Theorem. There exists a complete ordered field $\mathbf{R}$, called the *real numbers.* In fact, $\mathbf{R}$ is the unique[1] complete ordered field.

[1] up to relabeling of elements.