MATH 112 MIDTERM 1 PRACTICE QUESTIONS SOLUTIONS

1. Prove that for all $n \geq 0$
$$\sum_{k=0}^{n} 2^k(k+1) = 2^{n+1}n + 1.$$

SOLUTION: Proof by induction. We want to prove
$$\sum_{k=0}^{n} 2^k(k+1) = 2^{n+1}n + 1 \quad \text{for all } n \geq 0. \tag{1}$$
We have
$$\sum_{k=0}^{0} 2^k(k+1) = 2^0 \cdot 1 = 1,$$
thus (1) holds for $n = 1$ (base case). Now assume that
$$\sum_{k=0}^{n} 2^k(k+1) = 2^{n+1}n + 1 \quad \text{for some } n \geq 0. \tag{2}$$
Then we have
$$\begin{aligned}\sum_{k=0}^{n+1} 2^k(k+1) &= \sum_{k=0}^{n} 2^k(k+1) + 2^{n+1}(n+1+1) \\ &= 2^{n+1}n + 1 + 2^{n+1}(n+2) \qquad \text{using (2)} \\ &= 2^{n+1}(n+n+2) + 1 \\ &= 2^{n+2}(n+1) + 1\end{aligned}$$
which is (1) for $n+1$. This proves the induction step.

2. Let $A, B, C \subseteq U$ be sets. Prove the following
   (a) $(A \cap C) \setminus B = (A \setminus B) \cap (C \setminus B)$.
   (b) $U \setminus (A \setminus B) = (U \setminus A) \cup B$.

SOLUTION:

(a) $x \in (A \cap C) \setminus B \iff x \in A \cap C$ and $x \notin B \iff x \in A$ and $x \in C$ and $x \notin B$ $\iff x \in A$ and $x \notin B$, and $x \in C$ and $x \notin B \iff x \in A \setminus B$ and $x \in A \setminus C \iff$ $x \in (A \setminus B) \cap (C \setminus B)$.

(b) $x \in U \setminus (A \setminus B) \iff x \notin (A \setminus B) \iff x \notin A$ or $x \in B \iff x \in (U \setminus A)$ or $x \in B$ $\iff x \in (U \setminus A) \cup B$

3. Prove

$$\bigcup_{k\in\mathbb{N}}[-1/k,0]=[-1,0].$$

SOLUTION: Let $x\in\bigcup_{k\in\mathbb{N}}[-1/k,0]$, then $\exists k\in\mathbb{N}$ such that $x\in[-1/k,0]\implies -1/k\leq x\leq 0$ for some $k\in\mathbb{N}\implies -1\leq x\leq 0\implies x\in[-1,0]$. Thus $\bigcup_{k\in\mathbb{N}}[-1/k,0]\subseteq[-1,0]$. OTOH: let $x\in[-1,0]\implies x\in[-1/k,0]$ for $k=1\implies x\in\bigcup_{k\in\mathbb{N}}[-1/k,0]$. This implies $[-1,0]\subseteq\bigcup_{k\in\mathbb{N}}[-1/k,0]$.

4. Let R be a relation on $\mathbb{R}\times\mathbb{R}$ given by $(a,b)R(c,d)$ if $a-c$ and $b-d$ are integers.

(a) Prove that R is an equivalence relation.

SOLUTION: We have $a-a=0$ and $b-b=0$ and 0 is an integer, so $(a,b)R(a,b)$ which shows **reflexivity**. Now if $(a,b)R(c,d)$, then $a-c=k\in\mathbb{Z}$ and $b-d=n\in\mathbb{Z}$. Therefore, $c-a=-k\in\mathbb{Z}$ and $d-b=-n\in\mathbb{Z}$, so $(c,d)R(a,b)$ which shows **symmetry**. Finally, if $(a,b)R(c,d)$ and $(c,d)R(e,f)$ we have

$$a-c=k\in\mathbb{Z},\ b-d=n\in\mathbb{Z}$$
$$c-e=m\in\mathbb{Z},\ d-f=l\in\mathbb{Z}$$

Adding the above inequalities (first with third, and second with fourth) we get

$$a-e=k+m\in\mathbb{Z},\ \ b-f=n+l\in\mathbb{Z}$$

which implies $(a,b)R(e,f)$, and therefore we have **transitivity**.

(b) Prove that for any $(a,b)\in\mathbb{R}\times\mathbb{R}$ there exists $(c,d)\in[0,1)\times[0,1)$ such that $[(a,b)]=[(c,d)]$.

SOLUTION: Let $(x,y)\in[(a,b)]$, then $x=a+k$ and $y=b+n$ for some $k,n\in\mathbb{Z}$. Denote $\lfloor r\rfloor$ to be the greatest integer $\leq r$. Then for any $r\in\mathbb{R}$ we have $r-\lfloor r\rfloor\in[0,1)$, and

$$x=(a-\lfloor a\rfloor)+\lfloor a\rfloor+k=c+m,\quad\text{where } c=a-\lfloor a\rfloor\in[0,1),\ \text{and } m=\lfloor a\rfloor+k\in\mathbb{Z}$$
$$y=(b-\lfloor b\rfloor)+\lfloor b\rfloor+n=d+l,\quad\text{where } d=b-\lfloor b\rfloor\in[0,1),\ \text{and } l=\lfloor b\rfloor+n\in\mathbb{Z}$$

Therefore $(x,y)\in[(c,d)]$ with $c=a-\lfloor a\rfloor\in[0,1)$, and $d=b-\lfloor b\rfloor\in[0,1)$. OTOH: let $(x,y)\in[(c,d)]$ with c and d as before. Then

$$x=(a-\lfloor a\rfloor)+z,\ z\in\mathbb{Z}\implies x=a+(-\lfloor a\rfloor+z),\ -\lfloor a\rfloor+z\in\mathbb{Z}$$
$$y=(b-\lfloor b\rfloor)+p,\ p\in\mathbb{Z}\implies y=b+(-\lfloor b\rfloor+p),\ -\lfloor b\rfloor+p\in\mathbb{Z}$$

and thus $(x,y)\in[(a,b)]$.

5. Let $f\colon \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x + 2$, and $g\colon \mathbb{R} \to \mathbb{R}$ be defined by $g(x) = x^3$. Determine the following

   (a) What is the range of f? Of g?
   (b) Is f surjective, injective, bijective? Same questions for g.
   (c) If f or g is bijective, find the inverse.
   (d) Find $g \circ f$ and $f \circ g$.
   (e) What is the range of $g \circ f$? Of $f \circ g$?
   (f) Is $g \circ f$ surjective, injective, bijective? Same questions for $f \circ g$.

SOLUTION:

   (a) $Range(f) = Range(g) = \mathbb{R}$
   (b) f and g are both bijective
   (c) $f^{-1}\colon \mathbb{R} \to \mathbb{R}$, $f^{-1}(y) = y - 2$, and $g^{-1}\colon \mathbb{R} \to \mathbb{R}$, $g^{-1}(y) = y^{1/3}$
   (d) $g \circ f\colon \mathbb{R} \to \mathbb{R}$, $g \circ f(x) = g(x+2) = (x+2)^3$
   $f \circ g\colon \mathbb{R} \to \mathbb{R}$, $f \circ g(x) = f(x^3) = x^3 + 2$
   (e) $Range(g \circ f) = Range(f \circ g) = \mathbb{R}$
   (f) Both functions are bijective

6. Let F be an ordered field, $x, y \in F$. Prove the following

   (a) $-x - y = -(x + y)$.
   (b) $(-1)x = -x$.
   (c) $xy = 0$ iff $x = 0$ or $y = 0$.
   (d) If 0_1 and 0_2 are two additive identities, then $0_1 = 0_2$.

SOLUTION:

   (a) Recall that by definition $-(x+y)$ is the additive inverse of $(x+y)$, i.e. an element a such that $(x+y)+a = 0$. We therefore need to show that $(x+y)+(-x-y) = 0$, which would imply that $-x-y$ is the additive inverse of $x+y$, i.e. $-x-y = -(x+y)$. We have

$$\begin{aligned}(x+y) + (-x-y) &= (x-x) + (y-y) && \text{by associativity of addition}\\ &= 0 + 0 && \text{by additive inverses}\\ &= 0 && \text{by definition of } 0\end{aligned}$$

(b) Again, by definition $-x$ is the additive inverse of x so to prove $x + (-1)x = 0$. We have

$$\begin{aligned} x + (-1)x &= 1 \cdot x + (-1)x && \text{by definition of 1} \\ &= (1-1)x && \text{by distributivity} \\ &= 0 \cdot x && \text{by additive inverses} \\ &= 0 && \text{by property proved in class} \end{aligned}$$

7. Compute the following in $\mathbb{Z}_5$

(a) $\bar{7}^3 + \overline{38} - \overline{25} \cdot \overline{349}$

(b) $\bar{8}^{10}$

(Here, $\mathbb{Z}_5 = \mathbb{Z}/5\mathbb{Z}$ and $\bar{a} = [a]$, the equivalence class modulo 5. Your solutions should be representated by integers between 0 and 4.)

SOLUTION:

(a) $\bar{7}^3 + \overline{38} - \overline{25} \cdot \overline{349} = \bar{2}^3 + \bar{3} - \bar{0} \cdot \overline{349} = \bar{8} + \bar{3} - \bar{0} = \overline{11} = \bar{1}$

(b) $\bar{8}^{10} = \bar{3}^{10} = \left(\bar{3}^2\right)^5 = (\bar{9})^5 = \bar{4}^5 = \bar{4}^{2 \cdot 2} \cdot \bar{4} = \overline{16}^2 \cdot \bar{4} = \bar{1}^2 \cdot \bar{4} = \bar{4}$