Math 112 Homework for Friday, Week 12

1. Let $A \subset \mathbb{R}$, and fix $a \in A$. In this problem, we assume that a is a limit point of A and only consider continuous functions $f\colon A \to \mathbb{R}$ for which $f'(a)$ exists. You may use the following results (which all follow from the limit theorems):

   - If $\lim_{x\to a} h(x)$ and $\lim_{x\to a} k(x)$ exist, then $\lim_{x\to a} (-h(x)) = -\lim_{x\to a} h(x)$ and $\lim_{x\to a} (h(x)+k(x)) = \lim_{x\to a} h(x) + \lim_{x\to a} k(x)$.
   - Let c be a constant. Then $\lim_{x\to a} h(x) = c$ iff $\lim_{x\to a} (h(x) - c) = 0$.
   - Let c be a constant. Then $\lim_{x\to a} \frac{c}{x-a}$ exists iff $c = 0$.

   (a) Define a relation by

   $$f \sim g \quad \iff \quad \lim_{x\to a} \frac{f(x)-g(x)}{x-a} = 0.$$

   Prove that $\sim$ is an equivalence relation.

   (b) Given a function f, define a new function by $L_f(x) = f(a) + f'(a)(x-a)$. How is the graph of $L_f(x)$ related to the graph of $f(x)$? Illustrate by choosing a specific example of $f(x)$, then graphing f and L_f.

   (c) Prove that $f \sim L_f$.

   (d) Suppose that $L(x) = m(x-a) + b$ for some $m, b \in \mathbb{R}$. Prove that if $f \sim L$, then $L = L_f$. This result gives a precise meaning to the slogan "L_f is the best linear approximation of $f(x)$." (Hint: use part (c).)

2. For each of the following power series, determine the radius of convergence and justify your answer.

   (a) $\displaystyle\sum_{n=1}^{\infty} 3z^n$ (b) $\displaystyle\sum_{n=1}^{\infty} 3nz^n$ (c) $\displaystyle\sum_{n=1}^{\infty} \frac{(-5)^{-n}}{n} z^n$

   (d) $\displaystyle\sum_{n=1}^{\infty} \frac{3z^n}{n^3}$ (e) $\displaystyle\sum_{n=1}^{\infty} \frac{n^3}{n!} z^n$ (f) $\displaystyle\sum_{n=1}^{\infty} \frac{(3n)!}{(n!)^2} z^n$