Math 112 Homework for Friday, Week 10

Recall that your proofs should consist of complete sentences.

1. Give an example of divergent series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ such that $\sum_{n=1}^{\infty}(a_n + b_n)$ converges.

2. Prove the following series diverge using a theorem from this week.

   (a) $\displaystyle\sum_{n=1}^{\infty}(-1)^n$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2n+1}$.

3. Give an example of a series $\sum_{n=1}^{\infty} a_n$ such that $\lim_{n\to\infty} a_n = 0$, yet $\sum_{n=1}^{\infty} a_n$ diverges.

4. We've seen that if $r \in \mathbb{C}$ and $|r| < 1$, then $\displaystyle\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$. It follows that for all $a, r \in \mathbb{C}$ and $k \in \mathbb{Z}$, if $|r| < 1$, then

$$\begin{aligned}\sum_{n=k}^{\infty} ar^n &= ar^k + ar^{k+1} + ar^{k+2} + \dots \\ &= ar^k(1 + r + r^2 + \dots) \\ &= ar^k \sum_{n=0}^{\infty} r^n \\ &= \frac{ar^k}{1-r}.\end{aligned}$$

   Thus, $\displaystyle\sum_{n=k}^{\infty} ar^n = \frac{ar^k}{1-r}$. Use this formula to compute the following sums:

   (a) $\displaystyle\sum_{n=3}^{\infty}\left(\frac{1}{2}\right)^n$ (b) $\displaystyle\sum_{n=2}^{\infty} 7\left(-\frac{3}{4}\right)^n$ (c) $\displaystyle\sum_{n=1}^{\infty} 2\left(-\frac{i}{4}\right)^n$.

   Note: Express the solution to (c) in the form $a + bi$ where a and b are rational numbers.

5. Compute $\sum_{n=1}^{\infty} \frac{2}{n^2+2n}$ by first substituting $\frac{2}{n(n+2)} = \frac{1}{n} - \frac{1}{n+2}$, then using a telescoping series kind of argument. (It may help to write out several terms of the series after making the substitution.)

6. Let (a_n) be a sequence of nonnegative real numbers. Prove that $\sum_{n=1}^{\infty} a_n$ converges if and only if its sequence of partial sums is bounded.