Math 112 Homework for Friday, Week 9

1. The Fibonacci sequence $\{f_n\}$ is defined by $f_1 = f_2 = 1$ and the recursive relation $f_{n+2} = f_n + f_{n+1}$. The golden ratio is the number $\varphi := \frac{1+\sqrt{5}}{2}$. In this problem, you will show that the limit of the ratios f_{n+1}/f_n of the terms in the Fibonacci sequence is the golden ratio φ.

   (a) Verify that $-\varphi^{-1} = \frac{1-\sqrt{5}}{2}$, and that the solutions to the equation $x^2 - x - 1 = 0$ are $x = \varphi$ and $x = -\varphi^{-1}$.

   (b) Show that the golden ratio φ satisfies the equations

   $$\varphi^{n+2} = \varphi^n + \varphi^{n+1} \qquad \text{and} \qquad \varphi^{-n-2} = \varphi^{-n} - \varphi^{-n-1}.$$

   (c) Use induction to prove that for all $n \geq 1$, the n-th term of the Fibonacci sequence is

   $$f_n = \frac{1}{\sqrt{5}}\varphi^n - \frac{1}{\sqrt{5}}(-\varphi^{-1})^n.$$

   (Since the formula defining f_{n+2} involves two previous values of the Fibonacci sequence, your inductive hypothesis will consist of assumptions about f_n and f_{n+1} and your base case will concern the two initial values $n = 1, 2$.)

   (d) Prove that

   $$\lim_{n\to\infty} \frac{f_{n+1}}{f_n} = \varphi = \frac{1+\sqrt{5}}{2}.$$

2. In order to evaluate the expression

   $$\sqrt{3 + \sqrt{3 + \sqrt{3 + \cdots}}},$$

   define a sequence by $a_1 = \sqrt{3}$ and $a_{n+1} = \sqrt{3 + a_n}$, and then consider $\lim\limits_{n\to\infty} a_n$. Prove that the sequence $\{a_n\}$ is real, bounded, and increasing. Argue that its limit must exist, and then find the limit.