Math 112 Homework for Friday, Week 7

1. (Roots of unity) This problem considers the solutions to the equation $z^n = 1$ for integers $n > 0$ over the complex numbers. We will see below that there are exactly n solutions in $\mathbb{C}$ to this equation. They are called the *n-th roots of unity.*

   (a) For each of the cases $n = 2, 3, 4$: (i) find the n distinct solutions to $z^n = 1$ where $z \in \mathbb{C}$. (ii) find the polar form of each solution, and (iii) draw the solutions in the complex plane (a separate drawing for each case). (Hint: For the case $n = 3$, to solve $z^3 - 1 = 0$, note that $z - 1$ is a factor of $z^3 - 1$.)

   (b) Recall that we've shown $|zw| = |z||w|$ for all $z, w \in \mathbb{C}$. If $z^n = 1$, it follows that $|z|^n = |z^n| = |1| = 1$. Since $|z| \in \mathbb{R}$ and $|z| \geq 0$, this means that $|z| = 1$. So the length of a root of unity is always 1. What about the argument (angle)? Suppose that $z = \cos(\theta) + i\sin(\theta)$ is a root of unity. Think about what one could say about the argument of z^n (comparing with your examples to part (a)). Among all solutions to $z^n = 1$, choose the one with the smallest positive argument, θ. Describe θ in terms of n and then use this θ to describe the polar forms for all n solutions to $z^n = 1$.

2. Let D be a nonempty subset of $\mathbb{R}$. Let $f\colon D \to \mathbb{R}$ and $g\colon D \to R$ be bounded functions. Recall the notation

$$f(D) := \{f(x) : x \in D\} \quad \text{and} \quad g(D) := \{g(x) : x \in D\}.$$

   Define $h\colon D \to \mathbb{R}$ by $h(x) := f(x) + g(x)$. (This function h is usually denoted $f + g$, for obvious reasons). Suppose that $f(D)$ and $g(D)$ are bounded above (so their suprema exist by completeness of $\mathbb{R}$).

   (a) Show that $h(D)$ is bounded above. (Start: Let $y \in h(D)$. Therefore, $y = h(x)$ for some $x \in D$.)

   (b) Since $h(D)$ is bounded above, it has a supremum by completeness of $\mathbb{R}$. Show that $\sup h(D) \leq \sup f(D) + \sup g(D)$.

   (c) Find two specific functions $f, g : [0, 1] \to \mathbb{R}$ such that we have a strict inequality $\sup h([0, 1]) < \sup f([0, 1]) + \sup g([0, 1])$.