Math 112 Homework for Tuesday, Week 6

1. For each of the following subsets of $\mathbb{R}$, write down

$$\inf = \qquad \min = \qquad \sup = \qquad \max =$$

and fill in the correct values, using DNE if the quantity does not exist. You only need to give a justification if you think that it is necessary.

   (a) $(-1, 2) \cup [3, 4]$
   (b) $\left\{\frac{n}{n+1} : n \in \mathbb{N}\right\}$
   (c) $\bigcap_{n=1}^{\infty}(1 - 1/n, 1 + 1/n)$
   (d) $\{2^n/n : n \in \mathbb{N}^+\}$
   (e) $\left\{\sum_{k=1}^{n}(-1)^k \frac{1}{k} : n \in \mathbb{N}^+\right\}$

2. Mark each of the following statements as true or false. In each case, give a brief explanation (a few sentences) if it is true or a specific counterexample if it is false. Throughout, T denotes a nonempty subset of $\mathbb{R}$.

   (a) If T has an upper bound, then T has a least upper bound.
   (b) If T is bounded, then T has a maximum and a minimum.
   (c) If $T \subseteq \mathbb{Z}$ and T is bounded, then $\sup T \in \mathbb{Z}$.
   (d) If $T \subseteq \mathbb{Q}$ and T is bounded, then $\sup T \in \mathbb{Q}$.
   (e) If $m = \inf T$ and $m' < m$, then m' is a lower bound of T.
   (f) $\emptyset$ is bounded.
   (g) $\sup \emptyset$ and $\inf \emptyset$ do not exist.
   (h) If $S \subseteq T$ and S is nonempty, then $\inf T \leq \inf S$.

3. In this problem, you will prove that there are always elements of a set arbitrarily close to its supremum. More precisely: let T be a subset of an ordered field F, and suppose that $M := \sup T$ exists. Given an $\epsilon \in F$ with $\epsilon > 0$, prove that there exists $x \in T$ such that $M - x < \epsilon$. (Hint: First, carefully re-read the definition of the supremum given in the text. Second, consider $M - \epsilon$. Is it an upper bound for T?)