Math 112 Homework for Friday, Week 6

1. Recall that $\mathbb{R}$ is complete, meaning that every nonempty bounded above subset of $\mathbb{R}$ has a least upper bound. In this problem, you will show that the corresponding fact about the existence of greatest lower bounds is a consequence. First, we introduce some notation: given a subset $S \subseteq \mathbb{R}$, let $-S$ be the subset of $\mathbb{R}$ defined by $-S = \{-x : x \in S\}$.

   (a) Let $S \subseteq \mathbb{R}$, and suppose that M is the supremum of S. Prove that $-M$ is the infimum of $-S$.

   (b) Suppose that T is a nonempty subset of $\mathbb{R}$ that is bounded below. Prove that $\inf T$ exists in $\mathbb{R}$. (Hint: use part (a) and the completeness of $\mathbb{R}$.)

2. Write each complex number in the form $a + bi$, where $a, b \in \mathbb{R}$.

   (a) $(1 + 2i)(3 - 2i) + (2 - 5i)$

   (b) $(1 + i)^{-1}$

   (c) $(3 + 4i)(3 - 4i)$

   (d) $\dfrac{1 - 2i}{2 + 5i}$

3. Consider the set $\mathbb{R}^2$ with addition and multiplication defined by

$$(a, b) + (c, d) = (a + c, b + d) \qquad \text{and} \qquad (a, b) \cdot (c, d) = (ac, bd),$$

   respectively. Indicate which field axioms fail, giving a concrete counter-example in each case.

4. Let $x = \frac{\sqrt{3}}{2} + \frac{i}{2}$. Draw $x, x^2, x^3, x^4, x^5, x^6$, and x^7 in the complex plane. What do you observe?