Math 112 Homework for Tuesday, Week 5

1. Let F be a field. Prove that for any $x, y \in F$ the following equalities hold

$$(-x) \cdot y = -(x \cdot y)$$

2. In each of the examples below, list ALL field axioms that do not hold. Whenever possible, give a concrete example to illustrate why the axiom does not hold.

   (a) The integers, $\mathbb{Z}$, with ordinary addition and multiplication.

   (b) The nonnegative rational numbers, $\mathbb{Q}_{\geq 0}$, with ordinary addition and multiplication.

   (c) The set $F = \{x\}$, with just one element, and addition and multiplication defined as follows:

$$x + x = x, \quad x \cdot x = x.$$

3. Let $\mathbb{R}$ be the set of all real numbers, and define the new addition, $\oplus$ and the new multiplication, $\otimes$ on $\mathbb{R}$ as follows

$$x \oplus y = x + y + 2$$
$$x \otimes y = (x + 2)(y + 2) - 2.$$

   Prove that $\mathbb{R}$ is a field with these two binary operations. *Hint:* determine first the additive identity, $\mathbf{0}$, and multiplicative identity, $\mathbf{1}$. Then you can determine the additive and multiplicative inverses, $\ominus x$ and x^{-1}.

4. Let F be an ordered field and let $x \in F$ satisfy $x > 1$. Prove that for all positive integers n, we have $x^n > 1$.