MATH 111, SHEET 9: THE FUNDAMENTAL THEOREM OF CALCULUS

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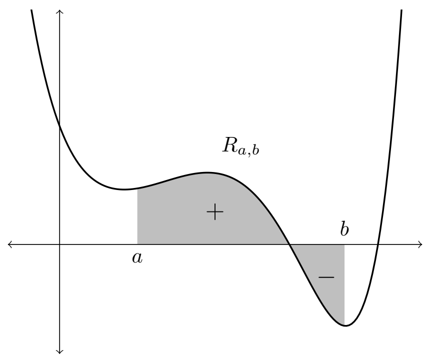
You are encouraged to read §5.1–5.4 of the textbook while studying this sheet.

In Exploration 7, we saw a pattern in our calculations of the area under a curve using Riemann sums. Recall that the integral of $f(x)$ from a to b is the *signed area* of the region $R_{a,b}$ in between the graph of $f(x)$ and the x-axis over the interval $[a, b]$.



$$\int_a^b f(x)\,dx = \text{area}(R_{a,b})$$

Signed area means the area above the x-axis *minus* the area below the x-axis. We found that the area between a and b under the graph of $f(x) = x^k$ for a positive integer k is

$$\int_a^b x^k\,dx = \frac{b^{k+1}}{k+1} - \frac{a^{k+1}}{k+1}$$

and that the area between a and b under the graph of $f(x) = \cos x$ is

$$\int_a^b \cos x\,dx = \sin b - \sin a.$$

Notice that the answers are given in terms of the values at a and b of a function *whose derivative* is the original function $f(x)$:

$$\frac{d}{dx}\frac{x^{k+1}}{k+1} = x^k \qquad \text{and} \qquad \frac{d}{dx}\sin x = \cos x.$$

In general, if $F(x)$ is a function whose derivative is $f(x)$, we call F an *anti-derivative* of f. The general statement of the pattern that we saw is:

The Fundamental Theorem of Calculus. Suppose that f is continuous on the interval $[a, b]$ and that F is an anti-derivative of f, i.e. $F'(x) = f(x)$. Then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

This is a remarkable theorem, and it ties together the two central ideas of this course: the slope of the tangent line (the derivative) and area, as approximated by sufficiently many boxes (the integral).

Suppose that G is another anti-derivative of f. Then

$$G'(x) = f(x) = F'(x), \qquad \text{and so} \qquad \frac{d}{dx}(G(x) - F(x)) = 0.$$

If the derivative of a function is zero, it must be a constant function (because the slope of the tangent line at all points is zero). Thus $G(x) - F(x) = K$ for some number K. This means that $G(x) = F(x) + K$, and so all of the anti-derivatives of $f(x)$ differ from each other by a constant. Notice that

$$G(b) - G(a) = (F(b) + K) - (F(a) + K) = F(b) - F(a),$$

so the right-hand side in the fundamental theorem of calculus *does not depend on the choice of anti-derivative of f*. This is a good thing, because the left-hand side also does not depend on the choice of anti-derivative!

Let's use the fundamental theorem to compute some more integrals. Remember that your answers give the signed area under the graph of the function, even though you no longer need Mathematica to do the computation!

Exercise 9.1.

(i) $\displaystyle\int_0^1 7x\,dx$

(ii) $\displaystyle\int_0^2 (x^2 + x^4)dx$

(iii) $\displaystyle\int_0^3 (x^3 - x)dx$

(iv) $\displaystyle\int_0^1 e^x\,dx$

(v) $\displaystyle\int_{-1}^3 (6x^2 - 3x + 4)dx$

(vi) $\displaystyle\int_{-1}^1 (x^{33} - x^{22} + x^{11})dx$

(vii) $\displaystyle\int_1^2 x^{-2}\,dx$

(viii) $\displaystyle\int_0^{\pi/4} \sin x\,dx$

(ix) $\displaystyle\int_{\pi/4}^{\pi/2} \sin x\,dx$

(x) $\displaystyle\int_0^{\pi/2} \sin x\,dx$

(xi) $\displaystyle\int_{-\pi/3}^{\pi/4} \cos x\,dx$

(xii) $\displaystyle\int_{-\pi/2}^{\pi/2} \sin x\,dx$

(xiii) $\displaystyle\int_{\pi/6}^{5\pi/6} \sin x\,dx$

(xiv) $\displaystyle\int_{\pi/6}^{5\pi/6} \cos x\,dx$

Exercise 9.2. Let a be an arbitrary positive number. What is $\int_{-a}^{a} \sin x \, dx$? More generally if $f(x)$ is an odd function, what is $\int_{-a}^{a} f(x)\, dx$? What can you say about this integral if f is an even function?

Here are some properties of the integral, which are illustrated by some of the calculations in Exercise 9.1.

Properties. (i) Integration respects sums of functions:

$$\int_a^b (f(x) + g(x))dx = \int_a^b f(x)\, dx + \int_a^b g(x)\, dx.$$

(ii) Integration respects multiplication by a constant:

$$\int_a^b cf(x)\, dx = c \int_a^b f(x)\, dx \qquad \text{for any number } c.$$

(iii) Integration over an interval $[a, c]$ can be split up into integration over subintervals $[a, b]$ and $[b, c]$:

$$\int_a^c f(x)\, dx = \int_a^b f(x)\, dx + \int_b^c f(x)\, dx.$$

(iv) Switching the bounds of integration introduces a minus sign:

$$\int_b^a f(x)\, dx = -\int_a^b f(x)\, dx$$

This last property is really a definition of the integral from b to a when $a < b$.[1]

The integral is a powerful tool. But it is good to be aware of the limits of its power. In particular, it is crucial in the fundamental theorem that $f(x)$ is continuous and that an anti-derivative $F(x)$ exists. In fact, some functions are not even integrable at all!

Exercise 9.3. Recall that a *rational number* is a fraction m/n where m and n are integers (and $n \neq 0$). Consider the function χ[2] defined by

$$\chi(x) = \begin{cases} 1 & \text{if } x \text{ is a rational number} \\ 0 & \text{if } x \text{ is not a rational number} \end{cases}$$

This strange function is called the *characteristic function of the rational numbers.* Try to think about the graph of χ. Then explain why χ is not integrable. In other words, show that $\int_0^1 \chi(x)\, dx$ does not exist—it cannot be well-approximated by upper and lower sums!

[1] The definition of the integral given in Exploration 7 can be generalized to include integration over intervals in the reverse order, as long as a minus sign is introduced into the width Δx of each box used in computing upper and lower sums. From this point of view, Property (iv) is true because of the introduction of this minus sign. In any case, it is important to note that *integration depends on the direction in which we integrate.*

[2] The symbol χ is the Greek letter *chi.*

Exercise 9.4. (i) Sketch the graphs of $f(x) = x^2 + 2$ and $g(x) = x + 4$, then find the area of the region in between the graphs.

(ii) Do the same for $f(x) = x^2 + 4x + 4$ and $g(x) = -x^2 + 4$.

Exercise 9.5. Consider the function

$$f(x) = \begin{cases} x + 4 & \text{when } x \leq 0 \\ 4 - x^2 & \text{when } 0 < x < 3 \\ x - 5 & \text{when } x \geq 3 \end{cases}$$

Sketch the graph of $f(x)$ and compute

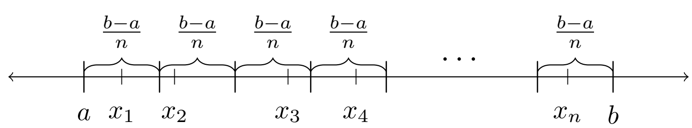
(i) $\displaystyle\int_{-4}^{0} f(x)\,dx$

(ii) $\displaystyle\int_{-4}^{2} f(x)\,dx$

(iii) $\displaystyle\int_{-4}^{3} f(x)\,dx$

(iv) $\displaystyle\int_{-4}^{5} f(x)\,dx$

The integral can also help us find the average value of a function f over the interval $[a, b]$. If we subdivide the interval into n equal width sub-intervals



and take a sample point x_i in each sub-interval, then the average value of $f(x)$ over the n points $x_1, \ldots, x_n$ is

$$\frac{f(x_1) + f(x_2) + \cdots + f(x_n)}{n} = \frac{f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x}{b - a},$$

where we have used that $\Delta x = (b - a)/n$. Notice that the numerator is a Riemann sum approximating the integral of $f(x)$ from $x = a$ to $x = b$. Taking the limit as $n \to \infty$, or equivalently as $\Delta x \to 0$, we find that

$$\text{the average value of } f(x) \text{ from } a \text{ to } b \;=\; \frac{1}{b - a}\int_a^b f(x)\,dx.$$

Exercise 9.6. The population of nutria in the canyon, as a function of the time t (in years) since they were introduced in 1930, is given by

$$P(t) = 20(1.049)^t$$

How many nutria were originally in the canyon in 1930? What is the current nutria population and rate of growth? What is the average population of canyon nutria over the last 87 years?

Exercise 9.7. The depth of snow (in inches) at Paradise is given as a function of time t (as a fraction of the year) by

$$S(t) = \begin{cases} 40 + 80\cos(2\pi t) & \text{when } t \leq 1/3 \text{ and } t \geq 2/3 \\ 0 & \text{when } 1/3 < t < 2/3 \end{cases}$$

Graph $S(t)$ explain why the definition depends on the cutoff values $t = 1/3$ and $t = 2/3$. What is the maximum and minimum depth of snow at Paradise? What is the average depth of snow at Paradise? (Bonus question: why is Paradise so snowy?)

Exercise 9.8. Let's compute some more integrals.

(i) $\displaystyle\int_0^{-2} x/2\,dx$

(ii) $\displaystyle\int_0^{-1} (4x^3 + 3x^4)dx$

(iii) $\displaystyle\int_1^{2} x^{-3}\,dx$

(iv) $\displaystyle\int_1^{2} x^{-1}\,dx$

(v) $\displaystyle\int_{-2}^{1} 2^x\,dx$

(vi) $\displaystyle\int_0^{1} \sqrt{1-x^2}\,dx$

(vii) $\displaystyle\int_9^{4} \sqrt{x}\,dx$

(viii) $\displaystyle\int_0^{\pi/4} \sec^2 x\,dx$

(ix) $\displaystyle\int_{\pi/6}^{\pi/2} (2\sin x - \cos x)\,dx$

(x) $\displaystyle\int_{\pi/3}^{\pi/2} (x^2 - \sin x)\,dx$

(xi) $\displaystyle\int_1^{5} 2xe^{x^2}\,dx$

(xii) $\displaystyle\int_0^{1} e^{-x^2}\,dx$

Exercise 9.9. Do the following problems from the textbook:

- §5.3: 20, 24, 28, 32, 38, 40, 41
- §5.4: 30, 40, 42, 43