MATH 111, SHEET 8: APPLICATIONS OF THE DERIVATIVE

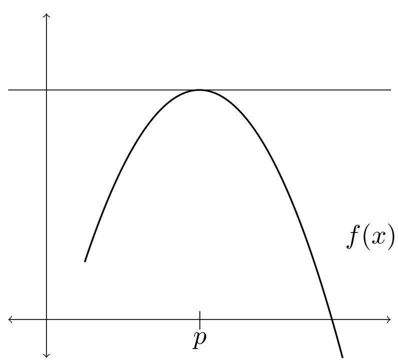
John Lind • October 25, 2017

We will now use our knowledge of the derivative to analyze *rates of change* and to solve *optimization problems*. **You are highly encouraged to read §4.1–4.3 and §4.6 of the textbook while studying this sheet.**

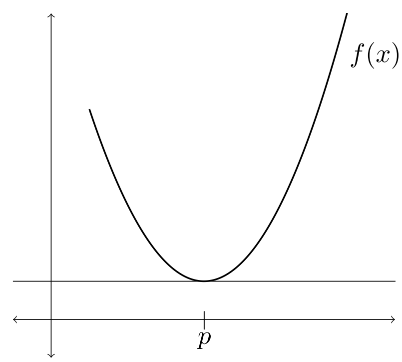
Suppose that $f(x)$ represents a quantity of interest, say the *profit margin* of our company, given some parameter x. We seek to maximize (or perhaps minimize) the output values $f(x)$ of the function f. In other words, we seek an x-value p such that

- p is a *local maximum*, meaning that $f(x) \leq f(p)$ for all x near p, or
- p is a *local minimum*, meaning that $f(p) \leq f(x)$ for all x near p.

If p is either a local max or a local min, then we say that p is a *local extremum*. Once we have found all of the local extema, we can plug them into the function $f(x)$, and determine which is the *global maximum*, meaning the point at which f has the largest value, and which is the *global minimum*, meaning the point at which f has the smallest value. The task of solving this problem is called *optimization*.



p is a local maximum



p is a local minimum

Notice that when p is a local extremum, the tangent line to f at p is flat, and thus $f'(p) = 0$.

Definition. We say that p is a *critical point* if $f'(p) = 0$.

We can summarize the pictures with the implication:

$$p \text{ is a local max or a local min} \quad \Longrightarrow \quad p \text{ is a critical point.}$$

It is very important to observe that the converse is *false*:

Exercise 8.1. Find an example of a function $f(x)$ and a critical point p for f that is not a local extremum.

This means that we must first find the critical points, and then determine which critical points are actually local extrema. In order to do this, examine the picture of the local maximum, and note that the slope of the tangent line to f at a point x to the left of p is positive, while at a point x to the right of p the slope of the tangent line is negative. In other words,

If $f'(x)$ changes from positive to negative values at $x = p$, then p is a local maximum.

Similarly, in the picture of the local minimum, the slopes of tangent lines to the left of p are negative, and the slopes of tangent lines to the right of p are positive. In other words,

If $f'(x)$ changes from negative to positive values at $x = p$, then p is a local minimum.

These two geometric observations about tangent lines are at the heart of our method of optimization.

Definition. The *second derivative* is obtained by taking the derivative of $f(x)$ with respect to x twice:

$$f''(x) = \frac{d}{dx}\Big(\frac{d}{dx}\big(f(x)\big)\Big).$$

Exercise 8.2. Find the second derivative of $f(x) = x^2 \cos x$

Just as $f'(a)$ is the slope of the tangent line to f at $x = a$, the value of the second derivative $f''(a)$ is the slope of the tangent line to $f'(x)$ at $x = a$. Therefore, the value $f''(p)$ measures the *rate of change of the derivative function* $f'(x)$. This means that if $f''(p)$ is negative, then $f'(x)$ is decreasing at p, and if $f''(p)$ is positive, then $f'(x)$ is increasing at p.

Exercise 8.3. Let $f(x) = x^2 \cos x$. Sketch the graphs of $f(x)$, $f'(x)$, and $f''(x)$ on the interval $[-2\pi, 2\pi]$. Indicate on the graph where $f'(x)$ is increasing, and where $f'(x)$ is decreasing. Confirm that this corresponds to where $f''(x)$ is positive and negative. What behavior in the graph of the original function $f(x)$ does this reflect?

Suppose that p is a critical point of $f(x)$, so $f'(p) = 0$. If $f''(p) < 0$, then $f'(x)$ is decreasing, hence changing from positive to negative values at $x = p$. Similarly, if $f''(p) > 0$, then $f'(x)$ is increasing at p, hence changing from negative to positive values at $x = p$. Combined with our observations above about local extrema, this gives:

The Second Derivative Test. Suppose that p is a critical point of f, i.e. $f'(p) = 0$.

- If $f''(p) < 0$, then p is a local maximum of f.
- If $f''(p) > 0$, then p is a local minimum of f.
- If $f''(p) = 0$, then the second derivative test gives no information.

Here is a summary of our method of optimization:

(1) Find the critical points of $f(x)$.

(2) For each critical point p, use the second derivative test to determine if p is a local maximum or a local minimum. If the second derivative test gives no information, then we must include p in step (3) when comparing values, or determine the behavior of f at p through other means.

(3) Compare the values $f(p)$ at the relevant local extrema to determine the global maximum and global minimum. If the domain of possible inputs of $f(x)$ is constrained, say to lie in some interval $a \leq x \leq b$, then we should also compare with the values of f at the endpoints of the constraint, in this case at the endpoints $f(a), f(b)$ of the interval.

Example. Let's find the minimum value of the function $f(x) = x^3 - 6x^2 + 9x + 4$ subject to the constraint $x \geq 0$. We must first find the critical points. Differentiating, we have $f'(x) = 3x^2 - 12x + 9$. We set the derivative equal to zero and solve for x:

$$\begin{aligned} f'(x) = 3x^2 - 12x + 9 &= 0 \\ 3(x-1)(x-3) &= 0 \\ x &= 1, 3 \end{aligned}$$

Thus, the critical points of f are $x = 1$ and $x = 3$. A this point, we could just plug our values into f, as well as the boundary condition $x = 0$ of the domain of definition $x \geq 0$, and compare the outputs to find the minimum. However, let's use the second derivative test for practice. The second derivative function is $f''(x) = 6x - 12$, and so the values at our critical points are:

$$\begin{aligned} f''(1) &= 6 - 12 = -6 < 0 \\ f''(3) &= 18 - 12 = 6 > 0 \end{aligned}$$

By the second derivative test, the critical point $x = 1$ is a local maximum, and the critical point $x = 3$ is a local minimum. Thus we only need to compare

$$f(3) = 4 \qquad \text{and} \qquad f(0) = 4.$$

Since they have the same output value, we conclude that the global minima of f occur at the points $x = 0$ and $x = 3$ with the value $f(0) = f(3) = 4$.

Exercise 8.4. Do the following problems from the textbook:

- §4.1: 28, 33, 44, 46
- §4.2: 4, 8, 10, 18, 19, 26, 28, 34
- §4.3: 18, 22, 30, 38, 42

The chain rule can help us understand how the rates of change of different quantities are related, even if we do not have explicit formulas for the quantities. This method of analysis is called *related rates.*

Example. Suppose that r is the radius of an expanding spherical balloon, and that the volume of the balloon is increasing at a constant rate of $2\,\text{cm}^3$ per second. Using the formula for the volume of a sphere of radius r

$$V = \frac{4}{3}\pi r^3,$$

we differentiate both sides with respect to the time variable t;

$$\frac{d}{dt}V = \frac{d}{dt}\Big(\frac{4}{3}\pi r^3\Big).$$

On the right, the constant $4\pi/3$ is unaffected by the differential operator, so we need to take the derivative of r^3. Since r depends on t, we need to apply the chain rule with the outside function $(-)^3$ and the inside function r:

$$\frac{dV}{dt} = \frac{4}{3}\pi \cdot 3r^2 \cdot \frac{dr}{dt} = 4\pi r^2 \cdot \frac{dr}{dt}$$

We are given that the volume is expanding at a rate of 2cm^3 per second, so we know that the left-hand side is $dV/dt = 2$. Solving the equation for dr/dt, we find that:

$$\frac{dr}{dt} = \frac{2}{4\pi r^2} = \frac{1}{2\pi r^2}.$$

This gives us a formula for the rate of change of r, *even though we do not have an explicit formula for $r(t)$ itself.* For example, when the radius of the balloon is $r = 1$ cm, we find that is it increasing at a rate of

$$\frac{dr}{dt} = \frac{1}{2\pi(1)^2} = \frac{1}{2\pi} = 0.1592 \text{ cm/sec}.$$

When the radius is $r = 3$ cm, it is increasing at a rate of:

$$\frac{dr}{dt} = \frac{1}{2\pi(3)^2} = \frac{1}{18\pi} = 0.0177 \text{ cm/sec}.$$

Exercise 8.5. Do the following problems from the textbook:

- §4.6: 12, 16, 20, 34, 36, 40, 44, 50, 52