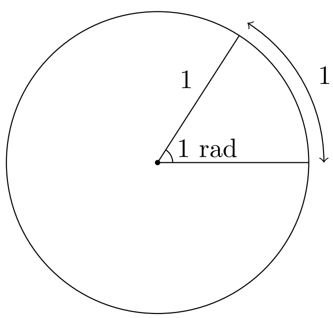
MATH 111, SHEET 7: TRIGONOMETRIC FUNCTIONS

John Lind • October 6, 2017

Before beginning this sheet, please read §1.5 and §3.5 of the textbook. Towards the end, §3.6 and §3.8 will be useful as well.

An angle of 1 *radian* is defined to be the angle at the center of a circle of radius 1 which determines an arc of length 1 along the perimeter of the circle. Since the circumference of the circle is 2π, this means that the angle which goes all the way around the circle is 2π radians. Therefore, 2π radians is equal to $360°$.

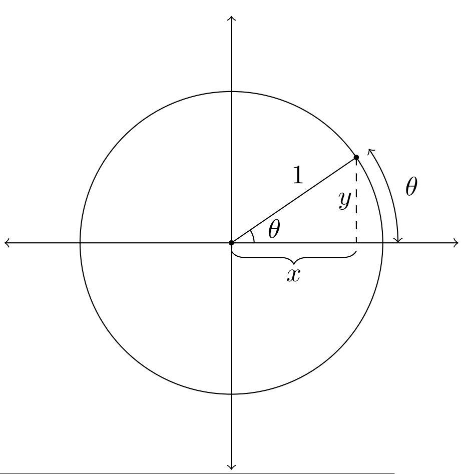


$$1 \text{ radian} = \frac{360°}{2\pi} = 57.2957795131\ldots°$$

$$1° = \frac{2\pi}{360} = 0.01745329251\ldots \text{ radians}$$

We will use radians to measure angles, unless we explicitly state otherwise. Suppose that θ[1] is an angle. The line at an angle θ from the x-axis determines a point (x, y) on the unit circle. Equivalently, the arc from $(1, 0)$ to (x, y) along the perimeter of the circle has length θ.

Definition. The *cosine* and *sine* of the angle θ are the x and y-coordinates of the point (x, y) on the unit circle determined by the angle θ.



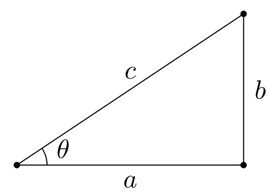
$$\boxed{\cos\theta = x \quad \text{and} \quad \sin\theta = y}$$

[1] The symbol θ is the Greek letter *theta*.

Exercise 7.1. Explain why $\cos^2\theta + \sin^2\theta = 1$. (We use the abbreviation $\cos^n\theta = (\cos\theta)^n$ for powers of trig functions.)

Question 7.2. A Ferris wheel has a radius of 50 meters. Let the origin $(0, 0)$ of the x, y coordinate axes be placed at the center of the Ferris wheel. If my compartment is at an angle θ from the x-axis, what are the coordinates (x, y) of my position?

Exercise 7.3. Given a right triangle with side-lengths a, b, and c as below, explain why $\cos\theta = a/c$ and $\sin\theta = b/c$.

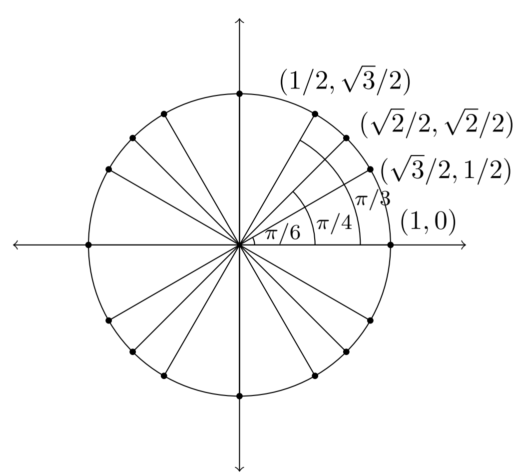


Definition. A function f is *even* if $f(-x) = f(x)$, and *odd* if $f(-x) = -f(x)$.

Exercise 7.4. Explain why $\cos\theta$ is an even function of θ. Explain why $\sin\theta$ is an odd function of θ.

Exercise 7.5. Complete the following table of values of cos and sin, using basic geometric reasoning and the given values. Then, draw a large unit circle and label all of the points $(\cos\theta, \sin\theta)$ for the given angles θ.

θ	$\cos\theta$	$\sin\theta$
0		
$\pi/6$	$\sqrt{3}/2$	$1/2$
$\pi/4$	$\sqrt{2}/2$	$\sqrt{2}/2$
$\pi/3$		$\sqrt{3}/2$
$\pi/2$		
$2\pi/3$	$-1/2$	
$3\pi/4$	$-\sqrt{2}/2$	
$5\pi/6$		
π	-1	0
$7\pi/6$		
$5\pi/4$		
$4\pi/3$		$-\sqrt{3}/2$
$3\pi/2$		-1
$5\pi/3$	$1/2$	
$7\pi/4$		
$11\pi/6$		
2π		



Exercise 7.6. Graph the functions $f(x) = \sin x$ and $g(x) = \cos x$. You should be able to see that sine is an odd function and that cosine is an even function from the graphs.

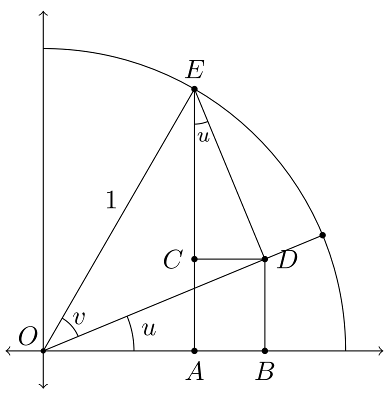
We've now come to our first theorem. A *theorem* is a true mathematical statement that is remarkable because of the beauty of its truth.[2]

Theorem. $\dfrac{d}{dx}(\sin x) = \cos x$

The fact that $\cos x$ is the derivative of $\sin x$ might be plausible because of the graphs of $\sin x$ and $\cos x$. We will give a careful proof of this fact using our previous calculations and a few trigonometric identities. We will need the sum formulas for sine and cosine:

$$\begin{aligned}\sin(u \pm v) &= \sin u \cos v \pm \cos u \sin v \\ \cos(u \pm v) &= \cos u \cos v \mp \sin u \sin v\end{aligned}$$

The use of "$\pm$" and "$\mp$" means that when the first is "+", the second is "−", and vice versa. Here is a proof of the first sum formula for sine.



$$\begin{aligned}\sin(u+v) = AE &= BD + CE \\ &= \frac{BD}{OD} \cdot OD + \frac{CE}{DE} \cdot DE \\ &= \sin u \cdot \cos v + \cos u \cdot \sin v\end{aligned}$$

Exercise 7.7. Replace v by $-v$ to derive the other sum formula for sine:

$$\sin(u - v) = \sin u \cos v - \cos u \sin v.$$

Subtracting the second sum formula for sine from the first, we find that

$$\begin{aligned}\sin(u+v) - \sin(u-v) &= (\sin u \cos v + \cos u \sin v) - (\sin u \cos v - \cos u \sin v) \\ &= 2 \cos u \sin v\end{aligned}$$

We will now make a change of coordinates so that this formula is easier to use. Let $A = u+v$ and $B = u - v$. Then $u = (A+B)/2$ and $v = (A-B)/2$, so the difference formula in terms of A and B is

$$\sin A - \sin B = 2 \cos\left(\frac{A+B}{2}\right) \sin\left(\frac{A-B}{2}\right).$$

Exercise 7.8. Use this difference formula in the definition of the derivative as a limit to compute the derivative of $f(x) = \sin x$. Congratulations, you have proved the theorem!

[2]As with all aspects of beauty, the determination of what qualifies as a theorem is up to the beholder.

A *corollary* is a statement that follows easily from a theorem.

Corollary. $\dfrac{d}{dx}(\cos x) = -\sin x$

Exercise 7.9. Use the definition of sine and cosine to explain why

$$\begin{aligned} \cos\theta &= \sin(\pi/2 - \theta) \\ \sin\theta &= \cos(\pi/2 - \theta) \end{aligned}$$

Exercise 7.10. Use the previous exercise to derive the corollary from the theorem. The chain rule will be crucial!

Definition. We define the tangent, secant, cotangent, and cosecant functions by:

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$
$$\sec\theta = \frac{1}{\cos\theta} \qquad \csc\theta = \frac{1}{\sin\theta}$$

Exercise 7.11. We say that a function f is *periodic* with period T if

$$f(x+T) = f(x), \quad \text{and } T \text{ is the smallest positive number with this property.}$$

In other words, T is the smallest time needed for the function to complete one cycle. What is the period of the sine and cosine functions? What is the period of the tangent function?

Exercise 7.12. Find the following derivatives:

(i) $\dfrac{d}{dx}\big(-\sin x\big)$

(ii) $\dfrac{d}{dx}\big(-\cos x\big)$

(iii) $\dfrac{d}{dx}\big(\tan x\big)$

(iv) $\dfrac{d}{dx}\big(\csc x\big)$

(v) $\dfrac{d}{dx}\big(\sec x\big)$

(vi) $\dfrac{d}{dx}\big(\cot x\big)$

Notice that there is a *fourfold periodicity* in the action of the differential operator d/dx on the sine and cosine functions:

$$\sin x \xmapsto{\frac{d}{dx}} \cos x \xmapsto{\frac{d}{dx}} (-\sin x) \xmapsto{\frac{d}{dx}} (-\cos x) \xmapsto{\frac{d}{dx}} \sin x$$

Exercise 7.13. Do the following problems from the textbook:

§3.5: 50, 56, 58, 60, 62

Exercise 7.14. Find the derivatives of the following functions

(i) $f(x) = \sin(x^5)$

(ii) $g(x) = \cos x \sin x$

(iii) $h(x) = \dfrac{\cos x}{x+1}$

(iv) $k(u) = e^{\cos u}$

(v) $w(x) = 3^{4\cos x}$

(vi) $\ell(x) = \ln(\sin(x^2))$

(vii) $\theta(y) = \dfrac{e^{y^2}}{\sin y}$

(viii) $t(w) = \cos(w^3 - 4^w)$

(ix) $p(x) = \tan(x^3)$

(x) $q(y) = \sin(\sin y))$

Definition. The function $\arcsin x$ is the inverse of the sine function. In other words,

$$\theta = \arcsin x \quad \Longleftrightarrow \quad \sin\theta = x$$

Similarly, $\arccos x$ and $\arctan x$ are the inverses of the cosine and tangent functions.

Question 7.15.

(i) What is $\cos(\arcsin x)$, in terms of x? Drawing a right triangle with hypotenuse 1 and side angle $\theta = \arcsin x$ may be helpful.

(ii) What is $\sin(\arccos x)$?

(iii) What is $\cos^2(\arctan x)$?

Exercise 7.16.

(i) Differentiate the equation $\sin(\arcsin(x)) = x$ to find $\dfrac{d}{dx}(\arcsin x)$.

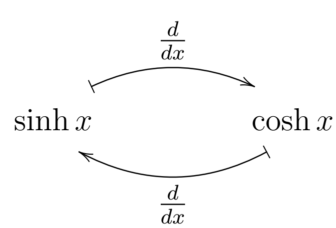
(ii) Apply a similar procedure to find $\dfrac{d}{dx}(\arccos x)$.

(iii) Find $\dfrac{d}{dx}(\arctan x)$.

Definition. The hyperbolic sine and cosine functions are defined by

$$\sinh x = \frac{e^x - e^{-x}}{2} \quad \text{and} \quad \cosh x = \frac{e^x + e^{-x}}{2}.$$

Exercise 7.17. Verify that the differential operator d/dx has a twofold periodicity in acting on $\sinh x$ and $\cosh x$:



Question 7.18. Is there a function which admits a "onefold" periodicity in the action of d/dx? In other words, is there a function $f(x)$ for which

$$\frac{d}{dx}(f(x)) = f(x)?$$

Challenge 7.19. Write down a function $f(x)$ which admits a threefold periodicity in the action of d/dx. In other words,

$$\frac{d^3}{dx^3}f(x) := \frac{d}{dx}\left(\frac{d}{dx}\left(\frac{d}{dx}(f(x))\right)\right) = f(x),$$

but this equation would not be true using fewer than three instances of d/dx. We can write this succinctly as

$$f^{(3)}(x) = f(x),$$

where $f^{(k)}(x)$ denotes the *k-fold* derivative of $f(x)$.