MATH 111, SHEET 6: THE CHAIN RULE

John Lind • September 27, 2017

Before beginning this sheet, please read §3.4 of the textbook.

The *composition* of two functions f and g is the function $f \circ g$ obtained by plugging the output of g into the function f:

$$(f \circ g)(x) = f(g(x)).$$

The chain rule describes the derivative of the composition in terms of the derivatives of the constituent functions f and g.

Proposition (The Chain Rule)**.** If f and g are differentiable functions, then

$$\boxed{(f \circ g)'(x) = f'(g(x)) \cdot g'(x)}$$

We can restate this in terms of the other notation for the derivative as

$$\frac{d}{dx}(f(g(x))) = \frac{d}{dy}(f(y))\Big|_{y=g(x)} \cdot \frac{d}{dx}(g(x))$$

which is a bit cumbersome. After setting $y = g(x)$ and $z = f(y)$, the chain rule can be summarized by the simple (but potentially ambiguous) formula

$$\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx}$$

Here is an example of how this works in action. Suppose that

$$f(y) = \sqrt{y} \qquad \text{and} \qquad g(x) = 4x^2 + x - 5.$$

The composition of f and g is the function

$$(f \circ g)(x) = f(g(x)) = \sqrt{4x^2 + x - 5}.$$

According to the chain rule, the derivative of the composition is

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x).$$

In order to use this formula we must first find the derivatives of f and g:

$$f'(y) = \frac{1}{2}y^{-1/2} = \frac{1}{2\sqrt{y}} \qquad \text{and} \qquad g'(x) = 8x + 1.$$

Plugging these into the chain rule, we see that

$$\begin{aligned} (f \circ g)'(x) &= \frac{1}{2\sqrt{4x^2 + x - 5}} \cdot (8x + 1) \\ &= \frac{8x + 1}{2\sqrt{4x^2 + x - 5}}. \end{aligned}$$

Exercise 6.1. Let $g(x) = x + 1$ and $f(y) = 1/y$. What is the composition $f \circ g$? Find its derivative $(f \circ g)'(x)$.

Exercise 6.2. Write $h(x) = \dfrac{3}{x^{55} + 3x^3}$ as the composition $f(g(x))$ of two functions f and g. Then use the chain rule to find $h'(x)$.

Exercise 6.3. Compute the derivative of $h(x) = (3x^3 + 4x^2 - 5x + 1)^5$ using the chain rule.

Exercise 6.4. Compute the derivative $\dfrac{d}{dx}\Big(\dfrac{9x-1}{x+3}\Big)$ using the chain rule.

The method of the last problem can be written down as a general rule:

Proposition (The quotient rule)**.** If $f(x) = \dfrac{g(x)}{h(x)}$, then

$$f'(x) = \frac{g'(x)h(x) - g(x)h'(x)}{(h(x))^2}.$$

Exercise 6.5. Derive the quotient rule from the product rule and the chain rule.

Exercise 6.6. Find the equation for the tangent line to $f(x) = \dfrac{2x^2+1}{x-3}$ at $x = 1$.

We can see why the chain rule is true by deducing it from a new axiom for tangency:

AXIOM (IX) If $g_1(x)$ is tangent to $g_2(x)$ at $x = a$ and $f_1(y)$ is tangent to $f_2(y)$ at $y = g(a)$, then $(f_1 \circ g_1)(x)$ is tangent to $(f_2 \circ g_2)(x)$ at $x = a$.

In other words,

$$g_1(x) \overset{x=a}{\top} g_2(x) \quad \text{and} \quad f_1(y) \overset{y=g(a)}{\top} f_2(y) \quad \text{imply that} \quad f_1(g_1(x)) \overset{x=a}{\top} f_2(g_2(x)).$$

We can summarize AXIOM (IX) by saying that tangency is preserved by composition of functions.

Exercise 6.7. Deduce the chain rule from AXIOM (IX).

Challenge 6.8. Show that AXIOM (IX) is satisfied by the definition of tangency from Sheet 5.

Intermission! You are strongly encouraged to consult §1.3 and §3.6 of the textbook while working on the rest of the sheet.

The *inverse function* of $f(x)$ is the function f^{-1} which satisfies

$$f^{-1}(y) = x \qquad \iff \qquad y = f(x).$$

In other words,

$$f^{-1}(f(x)) = x \qquad \text{and} \qquad f(f^{-1}(y)) = y.$$

Exercise 6.9. Explain why the inverse of $f(x) = x^2$ is $f^{-1}(y) = \sqrt{y}$. Since we may only take the square root of *non-negative* functions, this example shows that in order to define the inverse of a function, we must sometimes restrict the domain of the input.

Exercise 6.10. Using the previous example as a guide, explain why the graph of the inverse function f^{-1} is the reflection of the graph of f over the line $y = x$.

Suppose that we did not know the derivative of $g(x) = \sqrt{x}$. Since the inverse of g is the function $g^{-1}(x) = x^2$, we can try to find $g'(x)$ by applying the chain rule to the composition $g^{-1}(g(x)) = x$ and then solving for $g'(x)$. Here is how this works out. Since $x = (g(x))^2$, taking the derivative of both sides, and applying the chain rule on the right gives

$$\begin{aligned} 1 = \frac{d}{dx}(x) = \frac{d}{dx}\left[(g(x))^2\right] &= 2(g(x)) \cdot g'(x) \\ &= 2\sqrt{x} \cdot g'(x). \end{aligned}$$

Solving for $g'(x)$ by dividing both sides by $2\sqrt{x}$, we find that $g'(x) = \dfrac{1}{2\sqrt{x}}$, as expected.

Exercise 6.11. On Exploration 5, you will define the base e exponential function $f(x) = e^x$, which is its own derivative: $f'(x) = f(x)$. Its inverse function is the natural logarithm function $f^{-1}(x) = \ln x$. Use the equation

$$x = e^{\ln x}$$

and the chain rule to deduce that $\dfrac{d}{dx}(\ln x) = \dfrac{1}{x}$.

In general, this same method shows that the derivative of an inverse function is given by

$$\boxed{(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}}$$

Exercise 6.12. Compute the following derivatives.

(i) $\dfrac{d}{dx}\left(\dfrac{x^3 - x}{x^2 + x + 1}\right)$

(ii) $\dfrac{d}{dx}\left(\ln(2x^3 + 2x)\right)$

(iii) $\dfrac{d}{dx}\left(e^{x^2}\right)$

(iv) $\dfrac{d}{dx}\left(\dfrac{1}{\ln x}\right)$

(v) $\dfrac{d}{dx}\left(3^{4x^3}\right)$

(vi) $\dfrac{d}{dx}\left(\sqrt{e^{x+1}}\right)$

(vii) $\dfrac{d}{dx}\left(\dfrac{1}{\sqrt{\ln x}}\right)$