MATH 111, SHEET 5: THE DERIVATIVE AS A LIMIT

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In this sheet, we will describe the derivative using the language of limits. Along the way, we will answer the question from the end of Sheet 2 regarding a good definition of the notion of tangency. Let's start by recalling the definition of the limit from Exploration 5.

Definition. Suppose that $f(x)$ is a function, and that a and ℓ are numbers. We say that ℓ *is the limit of $f(x)$ as x approaches a* if for any positive amount of closeness ϵ, we can find some positive amount of closeness δ to a wherein x values have outputs $f(x)$ that are within ϵ of ℓ.[1] In other words,

$$\lim_{x\to a} f(x) = \ell \quad \text{MEANS:} \quad \begin{array}{c}\text{for any } \epsilon > 0, \text{ there exists } \delta > 0 \text{ such that} \\ \text{if } 0 < |x-a| < \delta, \quad \text{then } |f(x)-\ell| < \epsilon.\end{array}$$

An important point is that the value $f(a)$ of the function at $x = a$ does not matter in this definition. We never need to consider what happens when $x = a$, only values of x arbitrarily close to a. Remember that

$$\begin{array}{rl} |f(x)-\ell| < \epsilon & \quad \text{means that } f(x) \text{ is within } \epsilon \text{ of } \ell, \text{ and} \\ 0 < |x-a| < \delta & \quad \text{means that } x \text{ is within } \delta \text{ of } a \text{ and } x \neq a. \end{array}$$

It is a formal consequence of the definition that limits satisfy these basic properties:

Properties.

- Limits preserve addition. In other words,

$$\lim_{x\to a} (f(x) + g(x)) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x),$$

  provided that both limits on the right exist.

- Limits preserve multiplication. In other words,

$$\lim_{x\to a} (f(x) \cdot g(x)) = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x),$$

  provided that both limits on the right exist.

- A special case of the previous property is that limits preserve multiplication by a constant:

$$\lim_{x\to a} (c \cdot f(x)) = c \cdot \lim_{x\to a} f(x).$$

[1] The symbol δ is the Greek letter *delta*. We usually think of it as an extremely small quantity.

If $f(a) = \lim_{x \to a} f(x)$, then we say that f is *continuous at* $x = a$. If f is continuous everywhere that it is defined, then we say that f is *continuous.*

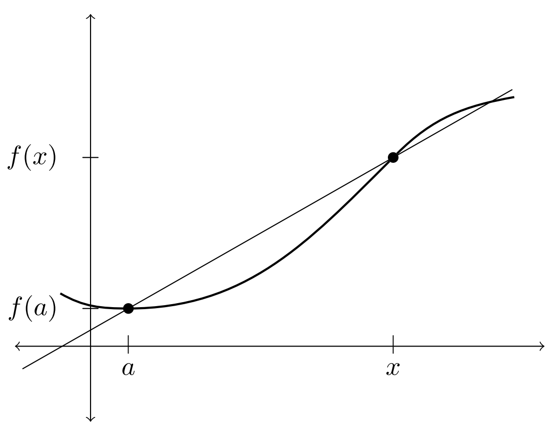
Exercise 5.1. Explain why the function $f(x) = x$ is continuous. Then explain why the function

$$f(x) = \begin{cases} x & \text{if } x \leq 0, \\ x+1 & \text{if } x > 0. \end{cases}$$

is not continuous at $x = 0$.

On Sheet 3, we defined the derivative $f'(a)$ to be the slope of the tangent line to $f(x)$ at $x = a$, provided that it exists. As we have discussed in class, we can approximate the slope of the tangent line by the slope of a line in between *two* points on the graph of f near the point $(a, f(a))$ that we are interested in. We will now use limits to make this idea of approximation precise.

A *secant line* to the graph of $f(x)$ is the line connecting two points of the form $(a, f(a))$ and $(x, f(x))$.



The slope of the secant line is

$$\frac{\Delta y}{\Delta x} = \frac{f(x) - f(a)}{x - a}.$$

Let's think of the point a as fixed and the point x as a variable. As we move the point x closer to the point a, the secant line draws closer to the tangent line to the graph of f at a. Expressed in terms of limits, this means that we may find the slope of the tangent line as a limit of the slopes of the secant lines. In other words, we have a new approach to the derivative!

Definition, Revisited. Suppose that the limit of the slopes of the secant lines to $f(x)$ as x approaches a exists. Then it is equal to the slope of the tangent line to $f(x)$ at $x = a$. In other words,

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

provided that the limit exists. In this case, we say that $f(x)$ is *differentiable at* $x = a$. This condition is equivalent to the existence of a tangent line to $f(x)$ at $x = a$.

Question 5.2. What is an example of a function that is not differentiable? Can you find an example of a function that is continuous but not differentiable?

In class, we decided that "the functions $f(x)$ and $g(x)$ are tangent at $x = a$" should mean that

1. $f(a) = g(a)$.
2. $f(x)$ and $g(x)$ have the same slope at $x = a$.

The first condition says that $f(x)$ and $g(x)$ intersect at $x = a$. For the second condition, we did not know before how to define slope, but now we can use the new definition of the derivative. In other words, $f(x)$ and $g(x)$ have the same slope at $x = a$ if $f'(a) = g'(a)$, and this is equivalent to the equation

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to a} \frac{g(x) - g(a)}{x - a}.$$

Let's move the right hand side to the left hand side and combine limits:

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} - \frac{g(x) - g(a)}{x - a} = 0.$$

If $f(a) = g(a)$, then those terms cancel when we combine fractions, leaving the equation

$$\lim_{x \to a} \frac{f(x) - g(x)}{x - a} = 0.$$

We finally have found a good definition of tangency.

Definition. The functions $f(x)$ and $g(x)$ are *tangent at* $x = a$ if f and g are continuous at $x = a$ and

$$\lim_{x \to a} \frac{f(x) - g(x)}{x - a} = 0.$$

All of our work on Sheet 3, including the sum rule and product rule for derivatives, was based on our axioms for the concept of tangency. We should check that this definition satisfies the axioms.

Exercise 5.3. Check that the definition of tangency satisfies axioms (I)–(VII) from Sheet 3. You are free to use the basic properties of limits mentioned at the beginning of the sheet.

Exercise 5.4. Let $l(x) = f'(a)(x - a) + f(a)$ be the tangent line to $f(x)$ at $x = a$. Use the definition of tangency to derive the equation for the derivative

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

As x approaches a, the distance $h = x - a$ between the points x and a gets very small. In other words, as $x \to a$, the distance $h \to 0$. Since $x = a + h$, we can rewrite the definition of the derivative in terms of a limit as $h \to 0$.

Definition, Revisited, Revisited. The derivative of $f(x)$ at $x = a$ is the number

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

The last formula doesn't use the variable x, so replacing a with x, we can define the derivative function, if it exists, by the formula

$$\boxed{f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}.$$

Exercise 5.5. Use this formula for the derivative to find the derivative function for each of the following functions.

(i) $f(x) = x$

(ii) $g(x) = x^2$

(iii) $p(x) = 1/x$

(iv) $q(x) = 1/x^2$

(v) $r(x) = 1/x^3$

(vi) $s(x) = \sqrt{x}$

There is an alternative notation for the derivative function:

$$\frac{d}{dx}(f(x)) = f'(x).$$

The nice thing about this notation is that we can think of $\frac{d}{dx}$ as an operator acting on the function $f(x)$. This makes the sum and product rules slightly easier to write down:

$$\text{Sum Rule:} \quad \boxed{\frac{d}{dx}\big(f(x) + g(x)\big) = \frac{d}{dx}(f(x)) + \frac{d}{dx}(g(x))}$$

$$\text{Product Rule:} \quad \boxed{\frac{d}{dx}\big(f(x) \cdot g(x)\big) = \frac{d}{dx}(f(x)) \cdot g(x) + f(x) \cdot \frac{d}{dx}(g(x))}$$

We can also write down the power rule using this notation:

$$\text{Power Rule:} \quad \text{for any integer } n, \boxed{\frac{d}{dx}x^n = nx^{n-1}}$$

We will soon see that the power rule holds when n is *any number*.

Remark. In the old days, people would write "dy/dx" for the derivative, and think of "dy" and "dx" as "infinitesimal amounts" of $y = f(x)$ and x, respectively. The way to make this idea precise is in terms of the definition of the derivative via limits that we have discussed. I recommend that you avoid this notation, since the derivative is not literally a fraction—it is only the *limit* of the fractions giving the slopes of the secant lines.