MATH 111, SHEET 3: THE DERIVATIVE

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In the last sheet, we explored examples of functions whose graphs are tangent to each other at a common point. Let us now describe this concept via a list of *axioms*. In other words, although we do not yet have a precise definition of the notion

"the functions $f(x)$ and $g(x)$ are tangent at $x = a$",

from now on we shall assume that it satisfies the following properties:

AXIOM (I) $f(x)$ is tangent to $f(x)$ at $x = a$.

AXIOM (II) If $f(x)$ is tangent to $g(x)$ at $x = a$, then $g(x)$ is tangent to $f(x)$ at $x = a$.

AXIOM (III) If $f(x)$ is tangent to $g(x)$ at $x = a$ and $g(x)$ is tangent to $h(x)$ at $x = a$, then $f(x)$ is tangent to $h(x)$ at $x = a$.

AXIOM (IV) If $f(x)$ is tangent to $g(x)$ at $x = a$, then $f(a) = g(a)$.

AXIOM (V) If two lines are tangent to each other at some point, then they must be equal.

AXIOM (VI) The parabola $f(x) = (x - a)^2$ is tangent to the horizontal line $g(x) = 0$ at $x = a$.

AXIOM (VII) Tangency is preserved by addition. In other words, if $f_1(x)$ is tangent to $g_1(x)$ at $x = a$, and $f_2(x)$ is tangent to $g_2(x)$ at $x = a$, then the sum $f_1(x) + f_2(x)$ is tangent to the sum $g_1(x) + g_2(x)$ at $x = a$.

AXIOM (VIII) Tangency is preserved by multiplication. In other words, if $f_1(x)$ is tangent to $g_1(x)$ at $x = a$, and $f_2(x)$ is tangent to $g_2(x)$ at $x = a$, then the product $f_1(x) \cdot f_2(x)$ is tangent to the product $g_1(x) \cdot g_2(x)$ at $x = a$.

Let us use the symbolic abbreviation "$f(x) \overset{x=a}{\top} g(x)$" for "the functions $f(x)$ and $g(x)$ are tangent at $x = a$". Notice that this concept, and the corresponding notation, both depend on the point a of tangency! Using this notation, the last axiom is:

AXIOM (VIII) If $f_1(x) \overset{x=a}{\top} g_1(x)$ and $f_2(x) \overset{x=a}{\top} g_2(x)$, then $f_1(x) \cdot f_2(x) \overset{x=a}{\top} g_1(x) \cdot g_2(x)$.

Exercise 3.1. Restate axioms (I) – (VII) using the $\overset{x=a}{\top}$ notation.

Axioms (I), (II) and (III) are rather obvious, but it is a good idea to insist that tangency follows such sensible rules. In mathematics, a property that satisfies rules like (I)–(III) is called an *equivalence relation*.

We will now develop the theory of the derivative, taking these axioms as our starting point. Your first task is to explain why the next proposition follows from the axioms.

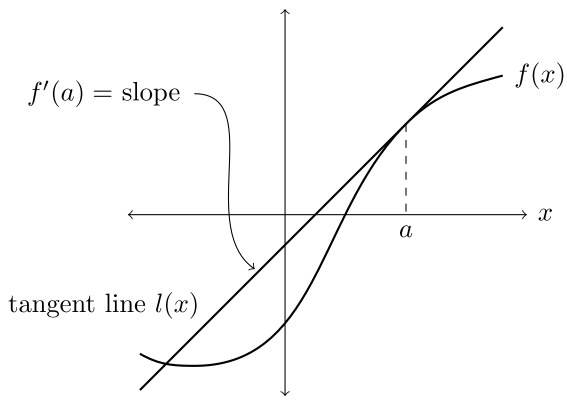
Proposition 3.2. *If a function has a tangent line at a given point, then the tangent line is unique. In other words, if $f(x)$ is tangent to both $l_1(x)$ and $l_2(x)$ at $x = a$, and the graphs of l_1 and l_2 are both lines, then $l_1(x) = l_2(x)$.*

Suppose that $f(x)$ is tangent to a line $l(x)$ at $x = a$. By Proposition 3.2, the tangent line $l(x)$ is unique. In particular, its slope m depends only on $f(x)$ and the choice of the value a.

Definition. The *derivative* of $f(x)$ at $x = a$ is the number

$$\boxed{f'(a) = \text{slope of the tangent line to } f(x) \text{ at } x = a.}$$

This definition requires the assumption that $f(x)$ has a tangent line at $x = a$.



Exercise 3.3. Let $f(x) = 4$. What is $f'(1)$? What is $f'(a)$ for an arbitrary number a?

Exercise 3.4. Let $f(x) = -2x + 5$. What is $f'(1)$? What is $f'(-2)$? What is $f'(a)$ for an arbitrary number a?

In order to reason correctly about the derivative it will be useful to first prove some logical stepping stones. Such a statement is traditionally called a *lemma.*

Lemma 3.5 (The first tangent line lemma)**.**

$$\textit{If } f(x) \overset{x=a}{\top} m(x-a) + c, \quad \textit{then } f(a) = c \textit{ and } f'(a) = m.$$

Proposition 3.6. *Suppose that $f(x) = mx + b$. Then for any number a, the derivative of f is $f'(a) = m$.*

Notice that when $m = 0$, this proposition says that the derivative of a constant function $f(x) = c$ is always zero!

Exercise 3.7. Let $f(x) = x^2$. Evaluate $f'(a)$ using these steps:

(i) Explain why $x^2 - 2ax + a^2 \overset{x=a}{\top} 0$.

(ii) Use this to explain why $x^2 \overset{x=a}{\top} 2a(x-a)+a^2$.

(iii) Deduce a formula for $f'(a)$.

The next statement is a reformulation of the definition of the derivative. You should briefly explain why it is true.

Lemma 3.8 (The second tangent line lemma). *Assuming that it exists, the tangent line $l(x)$ to $f(x)$ at $x=a$ can always be written in the form*

$$l(x) = f'(a)(x-a) + f(a).$$

The first and second tangent line lemmas are really useful for reasoning about derivatives. Use them to prove that the derivative of a sum is the sum of the derivatives:

Proposition 3.9 (The sum rule). *If $f(x) = g(x) + h(x)$, then $f'(a) = g'(a) + h'(a)$.*

Notice that this formula only makes sense if the tangent lines to g and h both exist at $x=a$, so the proposition implicitly makes this assumption.

A similar rule holds for the product of two functions. We will first need to prove another lemma.

Lemma 3.10 (The cancellation lemma).

$$\text{If } f(x) \overset{x=a}{\top} g(x) + h(x) \text{ and } h(x) \overset{x=a}{\top} 0, \quad \text{then } f(x) \overset{x=a}{\top} g(x).$$

Proposition 3.11 (The product rule). *If $f(x) = g(x)\cdot h(x)$, then*

$$f'(a) = g'(a)\cdot h(a) + g(a)\cdot h'(a).$$

Notice that this formula only makes sense if the tangent lines to g and h both exist at $x=a$, so the proposition implicitly makes this assumption.

Exercise 3.12. Let $f(x) = x^3$. What is $f'(a)$?

Exercise 3.13. Let $f(x) = x^4$. What is $f'(a)$?

Exercise 3.14. Let $f(x) = x^n$, where n is a positive integer. What is $f'(a)$?

Recall the slope function $S(a)$ from Sheet 2:

$$\begin{aligned} &\text{INPUT} \longrightarrow \text{OUTPUT} \\ &a \longmapsto S(a) = \text{slope of the tangent line to } f(x) \text{ at } x=a \end{aligned}$$

According our definition of the derivative, the value of the slope function S at a is exactly the derivative

$$\boxed{f'(a) = S(a)}\,!$$

Thinking of the derivative f' itself as a function, we can write its input using the variable name x instead of a, thus writing $f'(x)$ for the value of the derivative at input x. This is common practice, but please heed:

Warning. *The derivative function $f'(x)$ is different from the function*

$$l(x) = f'(a)(x-a) + f(a)$$

whose graph is the tangent line to $f(x)$ at $x = a$!

Exercise 3.15. Find the derivative function for each of the following functions.

(i) $f(x) = x^3 + x$

(ii) $g(x) = 99x^{100} - 3$

(iii) $h(x) = -12x^6 + 2x^2 - 5x$

(iv) $t(x) = 50x^3 - 8x^2 + 40x + 500$

(v) $w(x) = 5x^5 - 4x^4 + 3x^3 - 2x^2 + x - 1$

(vi) $z_1(x) = \frac{1}{2}x^2$

(vii) $z_2(x) = \frac{1}{3}x^3$

(viii) $z_3(x) = \frac{1}{4}x^4$

(ix) $z_n(x) = \frac{1}{n}x^n$ (where n is a positive integer)