MATH 111, SHEET 2: TANGENCY

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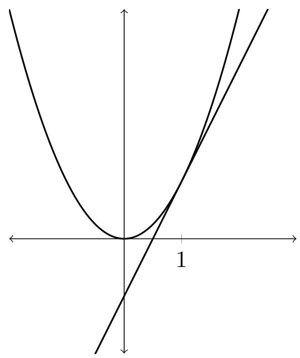
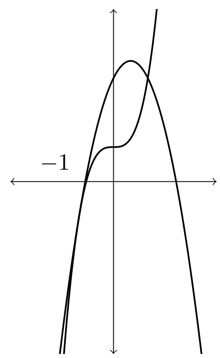
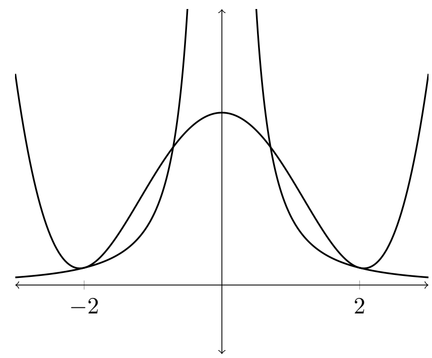
In this sheet, we will explore what it means for two graphs to be tangent to each other at a given point. The concept is best illustrated through examples.



$f(x) = x^2$
$g(x) = 2x - 1$
at $x = 1$



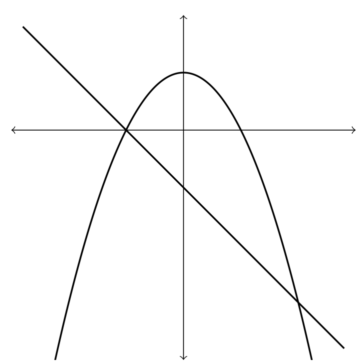
$f(x) = -2x^2 + 2x + 3$
$g(x) = 2x^3 + 1$
at $x = -1$



$f(x) = 1/x^2$
$g(x) = \frac{1}{8}x^4 - \frac{17}{16}x^2 + \frac{10}{4}$
at $x = -2$ and $x = 2$

In each example above, the graphs of the functions $f(x)$ and $g(x)$ are tangent to each other at the indicated point $x = a$. Intuitively, the graph of $f(x)$ is tangent to the graph of $g(x)$ at $x = a$ if their values draw closer to each other near $x = a$ in such a manner that the trajectories of the graphs align at $x = a$. Unfortunately, this is not a mathematically well-defined definition of tangency, but only an impressionistic description!

Here is a non-example: the graphs of $f(x) = 1 - x^2$ and $g(x) = -x - 1$ are not tangent to each other at any point.



$f(x) = 1 - x^2$
$g(x) = -x - 1$

We will allow ourselves to say simply "$f(x)$ and $g(x)$ are tangent at $x = a$" as shorthand for "the graph of $f(x)$ is tangent to the graph of $g(x)$ at $x = a$."

Exercise 2.1. Which of the following pairs of functions are tangent to each other at the specified point?

(i) $f(x) = x^2 + 1$ and $g(x) = 1$ at $x = 0$

(ii) $f(x) = x + 2$ and $g(x) = -2x - 1$ at $x = -1$

(iii) $f(x) = x + 2$ and $g(x) = x + 3$ at $x = -1$

(iv) $f(x) = x + 2$ and $g(x) = x + 2$ at $x = -1$

(v) $f(x) = 2x - 1$ and $g(x) = -x^2 + 4x - 4$ at $x = 1$

(vi) $f(x) = (x + 2)^2$ and $g(x) = -x^2$ at $x = -1$

(vii) $f(x) = x^2$ and $g(x) = -x^2 + 4x - 4$ at $x = 1$

(viii) $f(x) = x^2 - 2x$ and $g(x) = -1$ at $x = 1$

(ix) $f(x) = 2x^2 - 2x$ and $g(x) = 2x - 2$ at $x = 1$

Question 2.2. Can a pair of lines be tangent to each other at any point?

Question 2.3. We know already from the examples at the beginning of the sheet that $f_1(x) = x^2$ is tangent to $g_1(x) = 2x - 1$ at $x = 1$. Let $f_2(x) = x^2 - 2x$ and $g_2(x) = -1$. Notice that

$$f_1(x) + f_2(x) = 2x^2 - 2x \qquad \text{and} \qquad g_1(x) + g_2(x) = 2x - 2$$

Based on your answers to 2.1.(viii) and 2.1.(ix), is the tangency of f_1 to g_1 and the tangency of f_2 and g_2 related to the tangency of their sum? In general, if $f_1(x)$ is tangent to $g_1(x)$ at $x = a$, and $f_2(x)$ is tangent to $g_2(x)$ at $x = a$, does it follow that $f_1(x) + f_2(x)$ is tangent to $g_1(x) + g_2(x)$ at $x = a$? Additional examples might help you assess this question.

Question 2.4. We can ask a similar question regarding multiplication. If $f_1(x)$ is tangent to $g_1(x)$ at $x = a$, and $f_2(x)$ is tangent to $g_2(x)$ at $x = a$, does it follow that $f_1(x) \cdot f_2(x)$ is tangent to $g_1(x) \cdot g_2(x)$ at $x = a$? In order to answer this question, first consider the following examples:

(i) $f_1(x) = f_2(x) = x^2$ and $g_1 = g_2(x) = 2x - 1$ at $x = 1$

(ii) $f_1(x) = x$, $f_2(x) = x^2$, $g_1(x) = x$ and $g_2(x) = 0$ at $x = 0$

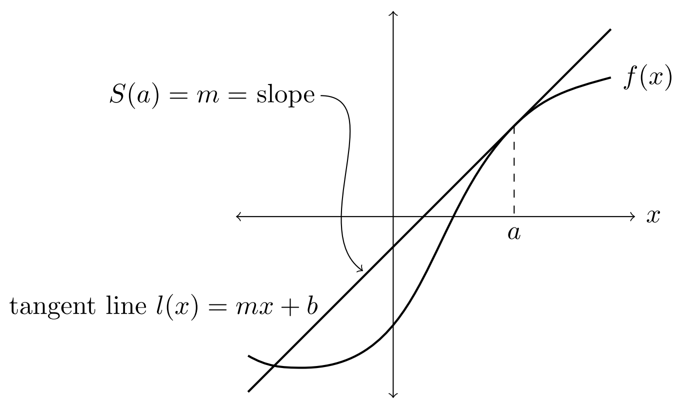
The case where a function $f(x)$ is tangent to a line $l(x) = mx + b$ at $x = a$ is particularly important. We then say that $l(x)$ *is a tangent line to* $f(x)$ *at* $x = a$.

Question 2.5. Can a function $f(x)$ have more than one tangent line at a point $x = a$? Why or why not?

Recall that a function is a rule that assigns to each input a corresponding output. We are interested in the *slope function* S that takes as input a number a and gives as output the slope of the tangent line to $f(x)$ at $x = a$:

$$\text{INPUT} \longrightarrow \text{OUTPUT}$$
$$a \longmapsto S(a) = \text{slope of the tangent line to } f(x) \text{ at } x = a$$

We might even call the slope of the tangent line to $f(x)$ at $x = a$ the "slope of $f(x)$ at $x = a$", but this is a mild abuse of terminology.



Exercise 2.6. Calculate the values of the slope-function $S(a)$ for each function below at a few values of a. Then graph the slope function, as a function of the variable a

(i) $f(x) = 7$

(ii) $g(x) = \frac{1}{2}x + 6$

(iii) $h(x) = x^2$

(iv) $t(x) = -x^2$

(v) $p(x) = \frac{1}{2}x^2$

(vi) $q(x) = x^3$

Of course, it doesn't matter what variable name we use to define a function, so we could equally well think of the slope function as a function of the variable x:

$$x \longmapsto S(x)$$

After all of our work thinking about tangency, let's ponder the important question that faces us.

Question 2.7. What is a good definition of the statement "$f(x)$ is tangent to $g(x)$ at $x = a$"?