MATH 111, SHEET 10: CONSTRUCTING ANTI-DERIVATIVES AND CALCULATING INTEGRALS

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You are encouraged to read §6.4 and §7.1–7.4 of the textbook while studying this sheet.

The fundamental theorem of calculus says that we can compute $\int_a^b f(x)\,dx$ by finding an anti-derivative $F(x)$ of $f(x)$, then evaluating:

$$\int_a^b f(x)\,dx = F(x)\Big|_{x=a}^{b} = F(b) - F(a).$$

But how do we find anti-derivatives? As we have seen, this is sometimes easy:

$$f(x) = x^5 \qquad \rightsquigarrow \qquad F(x) = \frac{1}{6}x^6$$

and sometimes hard:

$$f(x) = e^{-x^2} \qquad \rightsquigarrow \qquad F(x) = ???$$

On this sheet we will build a toolbox of techniques that construct anti-derivatives and calculate integrals.

Let's start by thinking about the anti-derivative of $f(x) = e^{-x^2}$. We are optimistic and so we assume that such an anti-derivative $F(x)$ exists. If this is true, then the fundamental theorem tells us that

$$F(b) - F(a) = \int_a^b e^{-t^2}\,dt$$

Let's think of the endpoint b of the interval as a variable x and set $a = 0$. Also, we will assume that $F(0) = 0$ (we know that adding a constant to an anti-derivative gives another anti-derivative, so we can always arrange for this to be the case). Then

$$F(x) = \int_0^x e^{-t^2}\,dt.$$

It turns out that this formula for $F(x)$ is indeed an anti-derivative of $f(x) = e^{-x^2}$. Unlike the other functions we have worked with, there is *not* a simpler formula for this anti-derivative—to evaluate $F(x)$ we must use Riemann sums to approximate the area under the curve.

This method of constructing anti-derivatives works in general. This idea is so important, that it is really a sequel to the fundamental theorem of calculus.

The Second Fundamental Theorem of Calculus (AKA the construction theorem for anti-derivatives)**.** If the function f is continuous and a is a number, then the function

$$F(x) = \int_a^x f(t)\,dt$$

is an anti-derivative of $f(x)$. In other words,

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x).$$

Exercise 10.1. For each of the following, graph the given function f, then graph the area-accumulation function

$$F(x) = \int_a^x f(t)\,dt$$

on the given interval $[a, b]$. In each case, is $F(x)$ an anti-derivative of $f(x)$?

(i) $f(x) = x - 2 \quad \text{on } [0, 4].$

(ii) $f(x) = |x - 2| \quad \text{on } [0, 4].$

(iii) $f(x) = \begin{cases} 1 & \text{when } 2 \le x \le 4, \\ -1 & \text{when } 4 \le x \le 6, \end{cases} \quad \text{on } [2, 6].$

(iv) $f(x) = \begin{cases} x - 2 & \text{when } 2 \le x \le 4, \\ x - 6 & \text{when } 4 \le x \le 6, \end{cases} \quad \text{on } [2, 6].$

Exercise 10.2. We might informally summarize the second fundamental theorem by the statement:

"*The rate of area accumulation under a curve at a given point is determined by the height of the curve at that point.*"

This is the central idea of differential calculus. It connects area accumulation with rate of change (i.e. the tangent line) by asserting that they are inverse operations. Give an informal proof of the second fundamental theorem by writing the derivative of the area accumulation function as a limit.

Exercise 10.3. Derive the first fundamental theorem from the second fundamental theorem.

Exercise 10.4. By the second fundamental theorem,

$$\frac{d}{dx}\int_1^x \frac{1}{t}\,dt = \frac{1}{x}.$$

But we already know an anti-derivative of $f(x) = 1/x$, namely $\ln|x|$. This suggests that

$$\ln x = \int_1^x \frac{1}{t}\,dt \quad (\text{for } x \geq 0).^1$$

Prove that this equation holds using the definition of the natural logarithm from Exploration 6 and the telescoping sub-interval decomposition of $[1, x]$ from Exploration 3.

The second fundamental theorem tells us that we can always use the integral to construct an anti-derivative of a continuous function. Motivated by this procedure, we will sometimes use the integral symbol without any bounds of integration to denote a general anti-derivative:

$$\int f(x)\,dx = \text{“an anti-derivative of } f(x)\text{”} = F(x) + K$$

The use of the generic constant K is meant to remind us that $f(x)$ has a *family* of anti-derivatives given by adding any constant to a single anti-derivative $F(x)$. Because of this notation, some people use the term “indefinite integral” for “anti-derivative”. I prefer to avoid this terminology, because *an anti-derivative is not the same thing as an integral.* The former is a function whose derivative is $f(x)$. The latter is the area under the graph of $f(x)$, as approximated by Riemann sums. But the notation can be useful, as long as we remember that

$$\int f(x)\,dx \qquad \text{means a very different thing than} \qquad \int_a^b f(x)\,dx.$$

Every rule for computing derivatives gives a corresponding rule for anti-derivatives. The chain rule

$$\frac{d}{dx}\Big(f(g(x))\Big) = f'(g(x)) \cdot g'(x)$$

says that $f(g(x))$ is an anti-derivative of $f'(g(x)) \cdot g'(x)$. In other words,

$$\int f'(g(x)) \cdot g'(x)\,dx = f(g(x)) + K.$$

This rule can be hard to use as is; it is often easier to set the inner function equal to a new variable u:

$$u := g(x), \qquad \text{so that } \frac{du}{dx} = g'(x), \text{ or “} du = g'(x)dx \text{”} .$$

This last equation isn't, strictly speaking, meaningful (what does “dx” really mean?!), but it allows us to deduce true statements so we accept it as a useful mnemonic. Substituting du for $g'(x)dx$ in the original expression transforms the task of finding an anti-derivative for $f'(g(x)) \cdot g'(x)$ into the simple task of finding an anti-derivative for $f'(u)$:

$$\int f'(g(x)) \cdot g'(x)\,dx = \int f'(u)\,du = f(u) + K = f(g(x)) + K.$$

Here's how this method, called *u-substitution*, works in practice.

[1] Often, this equation is taken as the *definition* of the natural logarithm!

Example. In order to find

$$\int x\sqrt{2+5x^2}\,dx$$

we observe that the "inner function" is $2+5x^2$, so we set $u := 2+5x^2$. Once we have made the choice for u, we find the corresponding expression for du:

$$\frac{du}{dx} = \frac{d}{dx}(2+5x^2) = 10x, \quad \text{and so} \quad du = 10x\,dx$$

In order to substitute u and du into the original function, we move the x over and multiply and divide by 10:

$$\int x\sqrt{2+5x^2}\,dx = \int \sqrt{2+5x^2}\cdot\frac{1}{10}\cdot 10x\,dx = \int \sqrt{u}\cdot\frac{1}{10}\,du$$

At this point in the calculation, it is crucial that all instances of x have disappeared and we are working entirely with the new variable u. Since we know the anti-derivative of the square-root function, we can evaluate and substitute back in for x:

$$\int \sqrt{u}\cdot\frac{1}{10}\,du = \frac{1}{10}\cdot\frac{2}{3}u^{3/2} + K = \frac{2}{30}(2+5x^2)^{3/2} + K.$$

Therefore, as can be checked by taking the derivative,

$$\int x\sqrt{2+5x^2}\,dx = \frac{2}{30}(2+5x^2)^{3/2} + K.$$

Exercise 10.5. Find the following anti-derivatives. You should check that your answer is correct by taking the derivative and comparing with the original function!

(i) $\displaystyle\int x^2\sqrt{9+x^3}\,dx$

(ii) $\displaystyle\int x^3\sqrt{4x^4+1}\,dx$

(iii) $\displaystyle\int x(x^2+5)^8\,dx$

(iv) $\displaystyle\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx$

(v) $\displaystyle\int \frac{e^x-e^{-x}}{e^x+e^{-x}}\,dx$

(vi) $\displaystyle\int \frac{e^{\sqrt{u}}}{\sqrt{u}}\,du$

(vii) $\displaystyle\int \frac{(\ln y)^2}{y}\,dy$

(viii) $\displaystyle\int \sin\theta(\cos\theta+2)^8\,d\theta$

(ix) $\displaystyle\int x\cosh x^2\,dx$

(x) $\displaystyle\int \sin^3 t\cos t\,dt$

(xi) $\displaystyle\int \frac{z}{1+4z^2}\,dz$

(xii) $\displaystyle\int \frac{x+2}{x^2+4x+10}\,dx$

Example. The u-substitution method can also be applied directly to integrals. To evaluate

$$\int_0^{\pi/3} \tan\theta \, d\theta$$

recall that $\tan\theta = \sin\theta/\cos\theta$. Set $u = \cos\theta$, so that $du = -\sin\theta d\theta$. We also need to find the bounds of integration in terms of u:

$$\begin{aligned} \theta = 0 \quad &\Longrightarrow \quad u = \cos 0 = 1 \\ \theta = \pi/3 \quad &\Longrightarrow \quad u = \cos(\pi/3) = 1/2 \end{aligned}$$

We now perform u-substitution in the original integral:

$$\int_0^{\pi/3} \tan\theta \, d\theta = \int_0^{\pi/3} \frac{\sin\theta d\theta}{\cos\theta} = \int_1^{1/2} \frac{-du}{u}$$

Notice that we made the substitution $\sin\theta \, d\theta = -du$ in the numerator, and that the bounds of integration go in the reverse of the usual orientation. To evaluate, we switch the order of the bounds of integration, introducing a minus sign, and compute in terms of the anti-derivative of $1/u$:

$$\int_1^{1/2} \frac{-du}{u} = -\int_{1/2}^{1} \frac{-du}{u} = \int_{1/2}^{1} \frac{du}{u} = \ln|u|\Big|_{u=1/2}^{1} = \ln(1) - \ln(1/2) = -\ln(1/2)$$

In this integral calculation, we didn't need to change the bounds of integration. Instead, we could first find the anti-derivative via u-substitution

$$\int \tan\theta \, d\theta = \int \frac{\sin\theta}{\cos\theta} \, d\theta = \int \frac{-du}{u} = -\ln|u| + K = -\ln|\cos\theta| + K,$$

and then apply the fundamental theorem to evaluate:

$$\int_0^{\pi/3} \tan\theta \, d\theta = -\ln|\cos\theta|\Big|_{\theta=0}^{\pi/3} = -\ln|\cos(\pi/3)| + \ln|\cos 0| = -\ln(1/2)$$

Exercise 10.6. Compute the following integrals

(i) $\displaystyle\int_0^{\pi/6} e^{-\cos\theta} \sin\theta \, d\theta$

(ii) $\displaystyle\int_0^{2} \frac{x}{(1+x^2)^2} \, dx$

(iii) $\displaystyle\int_1^{2} \frac{\cos\sqrt{x}}{\sqrt{x}} \, dx$

(iv) $\displaystyle\int_2^{3} xe^{x^2} \, dx$

(v) $\displaystyle\int_1^{3} \frac{\ln x}{x} \, dx$

(vi) $\displaystyle\int_0^{\pi/3} \frac{\sin\theta}{\cos^3\theta} \, d\theta$

(vii) $\displaystyle\int_0^4 \sqrt{y^2+y}(2y+1)\,dy$

(viii) $\displaystyle\int_0^{\pi/4} \sin^3(2\theta)\cos(2\theta)\,d\theta$

(ix) $\displaystyle\int_{-1}^4 \frac{1}{(x+2)^2}\,dx$

(x) $\displaystyle\int_{-\pi/2}^{\pi/2} \cos\theta\cos(\pi\sin\theta)\,d\theta$

(xi) $\displaystyle\int_0^{\pi} x^4\cos(x^5)\,dx$

(xii) $\displaystyle\int_1^2 t(9-2t^2)^{2/3}\,dt$

Remember, every rule about derivatives gives us a corresponding rule for anti-derivatives. The product rule

$$\frac{d}{dx}\big(f(x)\cdot g(x)\big) = f'(x)g(x) + f(x)g'(x)$$

says that $f(x)g(x)$ is an anti-derivative of $f'(x)g(x)+f(x)g'(x)$. In other words

$$f(x)g(x) = \int \big(f'(x)g(x)+f(x)g'(x)\big)\,dx = \int f'(x)g(x)\,dx + \int f(x)g'(x)\,dx$$

This is most useful if we move the first anti-derivative to the other side. In other words,

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx.$$

This formula is called *integration by parts.* If we set $u = f(x)$ and $v = g(x)$, we get

$$\boxed{\int udv = uv - \int vdu}$$

Example. For $\displaystyle\int_1^2 \ln x\,dx$, we set

$$u = \ln x \qquad dv = dx$$

which implies that

$$du = \frac{1}{x}dx \qquad v = x.$$

Applying integration by parts,

$$\int_1^2 \ln x\,dx = \ln x\cdot x\Big|_{x=1}^2 - \int_1^2 x\cdot\frac{1}{x}dx = 2\ln 2 - \int_1^2 dx = 2\ln 2 - 1.$$

Exercise 10.7. Compute the following anti-derivatives and integrals

(i) $\displaystyle\int xe^x\,dx$

(ii) $\displaystyle\int \ln(t^2)\,dt$

(iii) $\displaystyle\int x\sqrt[3]{2x+1}\,dx$

(iv) $\displaystyle\int x^2e^x\,dx$

(v) $\displaystyle\int \frac{z^7}{(4-z^4)^2}\,dz$

(vi) $\displaystyle\int \arctan(1/t)\,dt$

(vii) $\displaystyle\int_2^3 \frac{\ln(2x^5)}{x^2}\,dx$

(viii) $\displaystyle\int_{\pi/6}^{\pi/2} \theta\csc^2\theta\,d\theta$

(ix) $\displaystyle\int_1^5 \sqrt{2x}\ln x^3\,dx$

(x) $\displaystyle\int_0^1 x(x-1)^{11}\,dx$

(xi) $\displaystyle\int_0^5 xe^{2x}\,dx$

(xii) $\displaystyle\int_2^3 x\sin x\,dx$

Example. To find $\displaystyle\int \cos^2\theta\,d\theta$, set

$$u=\cos\theta \quad dv=\cos\theta d\theta,$$

so that

$$du=-\sin\theta d\theta \quad v=\sin\theta.$$

Then

$$\int \cos^2\theta\,d\theta=\int udv=uv-\int vdu=\cos\theta\sin\theta-\int-\sin^2\theta d\,\theta=\cos\theta\sin\theta+\int\sin^2\theta\,d\theta.$$

Substituting $\sin^2\theta=1-\cos^2\theta$ gives

$$\int \cos^2\theta\,d\theta=\cos\theta\sin\theta+\int(1-\cos^2\theta)\,d\theta=\cos\theta\sin\theta+\int d\theta-\int\cos^2\,d\theta.$$

Moving the anti-derivative of $\cos^2\theta$ at the end to the other side of the equation, we get

$$2\int\cos^2\theta=\cos\theta\sin\theta+\int d\theta=\cos\theta\sin\theta+\theta+K$$

and so

$$\int\cos^2\theta=\frac{\cos\theta\sin\theta}{2}+\frac{\theta}{2}+K.$$

Exercise 10.8. (i) Find $\displaystyle\int\sin^2\theta\,d\theta$

(ii) Use integration by parts twice to find $\int e^x \sin x\, dx$

(iii) Find $\int_0^\pi x^2 \cos(2x)\, dx$

(iv) Find $\int_0^1 x^3 \cos(5x)\, dx$

(v) Derive the formula

$$\int x^n \cos(ax)\, dx = \frac{1}{a} x^n \sin(ax) - \frac{n}{a} \int x^{n-1} \sin(ax)\, dx$$

(vi) Derive the formula

$$\int x^n \sin(ax)\, dx = -\frac{1}{a} x^n \cos(ax) + \frac{n}{a} \int x^{n-1} \cos(ax)\, dx$$

(vii) Derive the formula

$$\int \cos^n x\, dx = \frac{1}{n} \cos^{n-1} x \sin x + \frac{n-1}{n} \int \cos^{n-2} x\, dx$$

(viii) What is the analogous formula for $\int \sin^n x\, dx$?

Example. A *rational function* is a quotient of polynomials, such as $(x+3)/(x^2-4x+3)$. To find an anti-derivative, we factor the denominator and set the function equal to a sum of rational functions whose denominators are the linear terms:

$$\frac{x+3}{x^2-4x+3} = \frac{x+3}{(x-1)(x-3)} = \frac{A}{x-1} + \frac{B}{x-3}.$$

We need to solve for A and B. Multiplying by $(x-1)(x-3)$ gives

$$x + 3 = A(x-3) + B(x-1) = (A+B)x + (-3A - B).$$

Equating the coefficients of x and the constant terms, we have

$$\begin{cases} A + B = 1 \\ -3A - B = 3 \end{cases} \qquad \text{which has the solution} \qquad \begin{cases} A = -2 \\ B = 3 \end{cases}$$

Therefore,

$$\int \frac{x+3}{x^2-4x+3}\, dx = \int \frac{-2}{x-1}\, dx + \int \frac{3}{x-3}\, dx = -2\ln|x-1| + 3\ln|x-3| + K.$$

Exercise 10.9. Find the following anti-derivatives.

(i) $\displaystyle\int \frac{2x+4}{x^2+x}\,dx$

(ii) $\displaystyle\int \frac{1}{x^2+6x-16}\,dx$

(iii) $\displaystyle\int \frac{5x+3}{x^3-2x^2-3x}\,dx$

(iv) $\displaystyle\int \frac{6x^2-3x+1}{4x^3+x^2+4x+1}\,dx$

(v) $\displaystyle\int \frac{x^3}{x^2+x-2}\,dx$

(vi) $\displaystyle\int \frac{x^6+4x^3+4}{x^3-4x^2}\,dx$

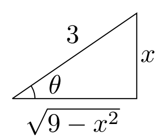
Recall from Sheet 7 that $\dfrac{d}{dx}(\arcsin x) = \dfrac{1}{\sqrt{1-x^2}}$ and $\dfrac{d}{dx}(\arctan x) = \dfrac{1}{1+x^2}$. This gives us two new anti-derivatives:

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + K \quad \text{and} \quad \int \frac{1}{1+x^2}\,dx = \arctan x + K.$$

There is a systematic way of finding anti-derivatives like these, called *trigonometric substitution*.

Example. To evaluate $\displaystyle\int \sqrt{9-x^2}\,dx$, we use the substitution $x = 3\sin\theta$, then draw a right triangle that relates x and θ:

$$\begin{cases} \sin\theta = \dfrac{x}{3} \\ \cos\theta = \dfrac{\sqrt{9-x^2}}{3} \end{cases}$$



To determine the formula for $\cos\theta$, we used the Pythagorean theorem to fill in the third side of the triangle. We also need to relate dx and $d\theta$ by taking the derivative of the original equation $x = 3\sin\theta$:

$$\frac{dx}{d\theta} = 3\cos\theta, \qquad \text{so } dx = 3\cos\theta\,d\theta$$

We now have all of the ingredients ready to make the substitution in the original anti-derivative and evaluate:

$$\int \sqrt{9-x^2}\,dx = \int 3\cos\theta \cdot 3\cos\theta\,d\theta = 9\int \cos^2\theta\,d\theta = \frac{9}{2}\cos\theta\sin\theta + \frac{9}{2}\theta + K.$$

In the final step, we used the anti-derivative of $\cos^2\theta$ that we found earlier using integration by parts.

In general, if other methods do not work and you want to try trig substitution,

if the integrand involves $\sqrt{a^2-x^2}$, try $x = a\sin\theta$.

The second rule for trig substitution is

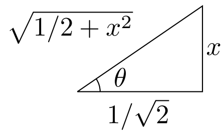
if the integrand involves a^2+x^2 or $\sqrt{a^2+x^2}$, try $x = a\tan\theta$.

Example. To find an anti-derivative of $1/(1+2x^2)$, we multiply and divide by $1/2$ to put the denominator in the form a^2+x^2:

$$\int \frac{1}{1+2x^2}\,dx = \int \frac{1/2}{(1/\sqrt{2})^2+x^2}\,dx$$

Now set $x = (1/\sqrt{2})\tan\theta$ and set up the triangle:

$$\tan\theta = \sqrt{2}x$$



Then we relate dx and $d\theta$ using the equation $x = (1/\sqrt{2})\tan\theta$

$$\frac{dx}{d\theta} = \frac{1}{\sqrt{2}}\frac{d}{d\theta}(\tan\theta) = \frac{1}{\sqrt{2}\cos^2\theta}, \qquad \text{so} \quad dx = \frac{d\theta}{\sqrt{2}\cos^2\theta}.$$

By the triangle, we know that

$$\cos\theta = \frac{1/\sqrt{2}}{\sqrt{1/2+x^2}} \qquad \text{so} \quad \cos^2\theta = \frac{1/2}{1/2+x^2}.$$

We are ready to substitute and compute the anti-derivative:

$$\begin{aligned}\int \frac{1}{1+2x^2}\,dx = \int \frac{1/2}{1/2+x^2}\,dx &= \int \cos^2\theta \cdot \frac{d\theta}{\sqrt{2}\cos^2\theta} \\ &= \frac{1}{\sqrt{2}}\int d\theta = \frac{1}{\sqrt{2}}\theta + K = \frac{1}{\sqrt{2}}\arctan(x\sqrt{2}) + K\end{aligned}$$

In the final step we used the relation $\theta = \arctan(x\sqrt{2})$ that comes from $x = (1/\sqrt{2})\tan\theta$.

Exercise 10.10. (i) Derive the equation $\int_0^1 \sqrt{1-x^2}\,dx = \frac{\pi}{4}$ from Exploration 4 using a trig substitution.

(ii) Find the area inside the ellipse $\frac{x^2}{4}+\frac{y^2}{9} = 1$.

(iii) $\int \frac{\sqrt{1-2x^2}}{x^2}\,dx$

(iv) $\int \frac{1}{x\sqrt{1+16x^2}}\,dx$

(v) $\int_0^1 \frac{1}{(3+4x^2)^{3/2}}\,dx$

(vi) $\int_0^{\sqrt{2}} \frac{x^3}{\sqrt{4-x^2}}\,dx$