MATH 111, SHEET 1: FUNCTIONS AND GRAPHS

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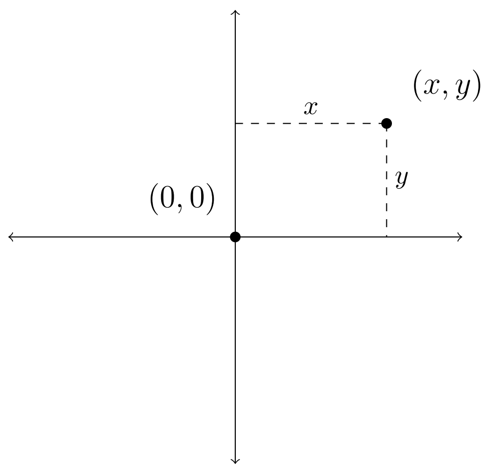
Let us begin with a careful introduction to functions and their graphs. The cartesian plane is the set of all pairs (x, y) of numbers. For example, $(0, 0)$ is a point on the cartesian plane called the origin and pictured below.



In general, a point (x, y) is drawn x units to the right and y units above the origin. The horizontal line is called the x-axis, and consists of those points (x, y) with $y = 0$. The vertical line is called the y-axis, and consists of those points (x, y) with $x = 0$.

Exercise 1.1. Draw the points $(3, 2), (-5, 3), (4.25, -1)$, and $(-\sqrt{2}, -\sqrt{2})$.

Notice that we are allowed to use any numbers x and y in describing points on the plane. We will not rigorously define what a "number" *is*[1]. We will use the following terminology for different types of numbers:

- the *natural numbers*: $0, 1, 2, 3, 4, 5, \ldots$
- the *integers*: $\ldots, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, \ldots$
- the *rational numbers*: fractions of the form m/n, where m and n are integers and $n \neq 0$
- the *real numbers*: all numbers that may be written in decimal notation, whether finite or infinite. For example, 1.02 and $2.\overline{333}\ldots$. This includes all sorts of numbers, such as 0, $\sqrt{5}$, π, $1/3$, $\sin(-300)$, e, and $2^{2^{2^2}}$.

In this course, when we use the term "number" without qualification, we mean a real number.

[1] But if you take Math 112 you will discuss such things and much more!

Definition. A *function* is a rule that assigns to each input number x an output number $f(x)$.

$$\text{INPUT} \longrightarrow \text{OUTPUT}$$
$$x \longmapsto f(x)$$

For example, the function $f(x) = 3x + 1$ takes the following values on the given inputs:

$$\text{INPUT} \longrightarrow \text{OUTPUT}$$
$$0 \longmapsto f(0) = 3 \cdot 0 + 1 = 1$$
$$1 \longmapsto f(1) = 3 \cdot 1 + 1 = 4$$
$$-\tfrac{4}{5} \longmapsto f(-\tfrac{4}{5}) = 3 \cdot (-\tfrac{4}{5}) + 1 = -\tfrac{7}{5}$$

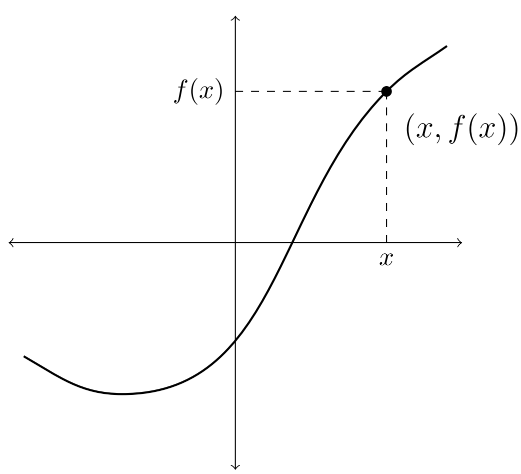
The same data may be displayed in a table of inputs and outputs:

x	$f(x)$
0	1
1	4
$-\frac{4}{5}$	$-\frac{7}{5}$

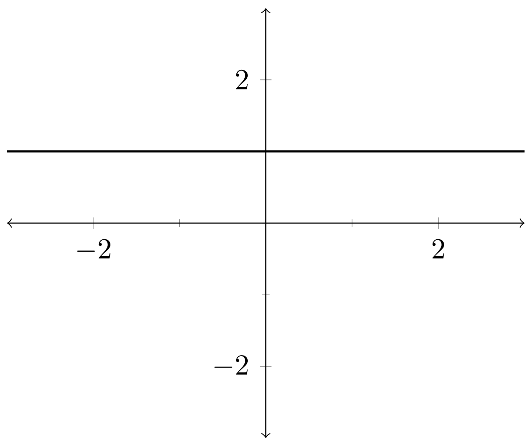
It would be impossible to write down every input and output value for this function because that would require writing down an infinite list of numbers. But the formula $f(x) = 3x+1$ tells us what the function *is* because we can use it to compute the effect of the function on any given input. Notice that it doesn't really matter that we used the variable name "x" in this formula. The expressions $f(x) = 3x + 1$, $f(y) = 3y + 1$ and $f(t) = 3t + 1$ all define the exact same function f.

It is important to remember that for a given function, we may or may not have a good understanding of the inner workings of the process that takes an input to its output. We may have a concrete formula for a function, such as $g(x) = \sqrt{x} - x^2$, or the function may be described in a more mysterious way.

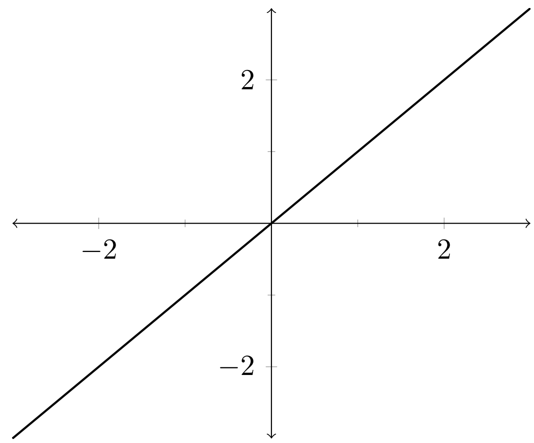
Definition. The *graph* of a function f is the set of all points $(x, f(x))$ whose first coordinate is an input x of f and whose second coordinate is the corresponding output $f(x)$.



For example, here are the graphs of the constant function $f(x) = 1$ and the function $g(x) = x$:



The graph of $f(x) = 1$



The graph of $g(x) = x$

We can also refer to the graph of a function f by the equation $y = f(x)$. This is shorthand for "the set of points (x, y) in the plane satisfying the equation $y = f(x)$".

Exercise 1.2. Sketch the graphs of:

(i) $f(x) = -2x + 3$

(ii) $g(x) = 7(x - 1)$

(iii) $h(x) = -\frac{2}{3}x + 6$

Exercise 1.3. Sketch the graphs of:

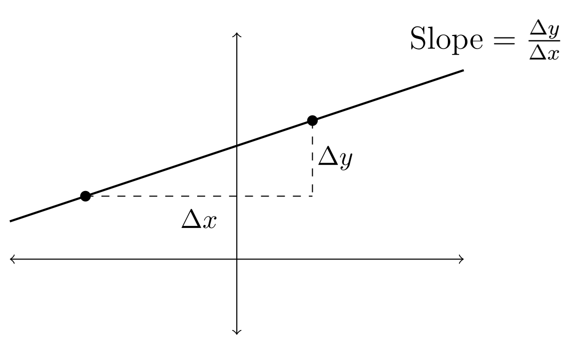
(i) $a(x) = x^2$

(ii) $b(x) = (x - 1)(x + 1)$

(iii) $c(x) = -2x^2 + 6x - 4$

Exercise 1.4. Where does the graph of the function $f(x) = -2x + 3$ intersect the graph of the function $a(x) = x^2$? What does this mean in terms of inputs and outputs?

Definition. The graphs in Exercise 1.2 are all *lines*. The *slope* of a line is the ratio of the vertical distance to the horizontal distance travelled over any segment of the line.



The Greek letter Δ (called "delta") stands for "change in".

Reality Check. Convince yourself that the slope of a line is the same regardless of which segment of the line is used to measure it. Mathematically, this means that the slope of a line is *well-defined.*

Exercise 1.5. Suppose that I ride my bicycle in a straight path away from home in such a manner that my distance from home (in meters) is given as a function of time (in seconds) by the function $d(t) = 5t + 10$. Graph the function $d(t)$ and explain the physical significance of the horizontal and vertical axes. What is my average speed from $t = 0$ to $t = 10$?

If the graph $y = f(x)$ is a line, then its slope is the *rate of change* of the quantity $f(x)$ as the quantity x varies:

$$\text{Slope} = \text{rate of change of } f(x) = \frac{\text{change in output values } f(x)}{\text{change in input values } x} = \frac{\Delta f(x)}{\Delta x}$$

Exercise 1.6. Find the slope of each line that you sketched in Exercise 1.2.

Question 1.7. What does the number m tell us about the graph of $f(x) = mx + b$? What does the number b tell us about the graph? If we factor $f(x)$ into the form $f(x) = m(x-a)+c$, what does the number a tell us about the graph?

Question 1.8. If the graph of a function f is a line, must the function be of the form $f(x) = mx + b$?

Question 1.9. How many parameters are needed to determine a line? What does "the space of all lines" look like?

The graphs in Exercise 1.3 are all *parabolas.* We call a function whose graph is a parabola a *quadratic function.*

Question 1.10. What is the slope of $y = x^2$? Does the slope of a parabola make sense?

Question 1.11. How many parameters are needed to determine a parabola? What does "the space of all parabolas" look like?