MATH 111, EXPLORATION 9

Due Wednesday, December 6*

A *differential equation* is an equation that relates functions and the derivatives of functions. For example, the differential equation

$$\frac{dy}{dx} = 4\sin x + 3\cos x$$

relates the derivative of y with respect to x and various functions of x. We think of the variable y as an unknown function that we wish to solve for. In this case, the differential equation can be solved by finding the anti-derivative of the right hand side:

$$y = \int dy = \int (4\sin x + 3\cos x)dx = 4\cos x - 3\sin x + K$$

It is important to include the generic constant K in the anti-derivative here, since we want to find the most general solutions possible for $y = f(x)$. If we impose the further *initial condition* $(x, y) = (0, 0)$, which corresponds to the requirement that $f(0) = 0$, then we can find a specific solution by solving for K;

$$0 = f(0) = 4\cos 0 - 3\sin 0 + K = 4 + K \qquad \Longrightarrow \qquad K = -4.$$

In summary, we have found that the solution of the differential equation $dy/dx = 4\sin x + 3\cos x$ with initial condition $(x, y) = (0, 0)$ is $y = 4\cos x - 3\sin x - 4$.

In the next example, the differential equation relates the derivative dy/dx and a function of the variable y (instead of x):

$$\frac{dy}{dx} = y^2$$

To solve this equation, we multiply both sides by dx/y^2 to get $dy/y^2 = dx$. While this equation might not have a rigorous meaning in terms of infinitesimal quantities, we can take anti-derivatives to get a meaningful equation relating y and x:

$$-\frac{1}{y} = \int \frac{dy}{y^2} = \int dx = x + K.$$

Again, we must include a generic constant to be guaranteed the most general set of solutions. Solving for y as a function of x, we find that the general solution to the differential equation $dy/dx = y^2$ is $y = -1/(x + K)$. By solving for K, you can find that the initial condition $(x, y) = (3, 1)$ is satisfied by the specific solution $y = -1/(x - 4)$.

If a differential equation sets dy/dx equal to the product of a function of x with a function of y, such as

$$\frac{dy}{dx} = (x^3 - x^9)\cos^2 y,$$

then we can use *separation of variables* to write an equation where one side only involves dy and y, while the other only involves dx and x:

$$\frac{dy}{\cos^2 y} = (x^3 - x^9)dx.$$

We then take anti-derivatives

$$\tan y = \int \frac{dy}{\cos^2 y} = \int (x^3 - x^9)\, dx = \frac{1}{4}x^4 - \frac{1}{10}x^{10} + K$$

and solve for y to find the general solution:

$$y = \arctan\big(\frac{1}{4}x^4 - \frac{1}{10}x^{10} + K\big)$$

Here are some problems to practice these techniques:

(1) Find the solution of the differential equation

$$\frac{dy}{dx} = 5^x - 5 \qquad \text{with initial condition } (x, y) = (1, -5).$$

(2) Find the solution of the differential equation

$$\frac{dy}{dx} = 3y \qquad \text{with initial condition } (x, y) = (0, 5).$$

More generally, find the solution of the differential equation

$$\frac{dy}{dx} = \alpha y \qquad \text{with initial condition } (x, y) = (0, \beta),$$

where α and β are arbitrary constants.[1]

(3) Use separation of variables to find the solution of the differential equation

$$\frac{dy}{dx} = 2xy - 6x \qquad \text{with initial condition } (x, y) = (2, 4).$$

For the rest of the exploration, we will consider the *Lotka–Volterra equations*

$$\begin{cases} \dfrac{dx}{dt} = \alpha x - \beta xy \\ \\ \dfrac{dy}{dt} = \delta xy - \gamma y \end{cases}$$

[1]The symbols α and β are the Greek letters *alpha* and *beta*.

where $\alpha, \beta, \gamma, \delta > 0$ are positive constants.[2] This pair of differential equations is important in biology and economics because it models the fluctuation of populations that are interdependent. The classic interpretation is in terms of predator/prey dynamics, where

$$x = p(t) = \text{the population of } \textit{prey} \text{ (say, bunny rabbits) at time } t$$

and

$$y = q(t) = \text{the population of } \textit{predators} \text{ (say, coyotes) at time } t$$

We might attempt to use separation of variables in the first equation to relate $x = p(t)$ and $y = q(t)$:

$$\ln x = \int \frac{dx}{x} = \int (\alpha - \beta y)\, dt = \int (\alpha - \beta q(t))\, dt = ???$$

But, we can't find a formula for the anti-derivative on the right side, since the integrand involves the unknown function $y = q(t)$. To solve either equation analytically requires that a solution of the other equation is already known. We are at an impasse! It turns out that there is no way to write down analytic solutions to the Lotka-Volterra equations. Instead, we will use Mathematica to find *numerical solutions*.

Let's concentrate on the case $\alpha = \delta = 1$, $\beta = \gamma = 2$ for now. If we plot an arrow in the xy-plane at the point (x, y) which points in the direction $(p'(t), q'(t)) = (x - 2xy, xy - 2y)$, then we may visualize the Lotka-Volterra equations as a *vector field*. Do this in Mathematica using the command

`StreamPlot[ {x - 2x*y, x*y - 2y},` $\{x, x_{\min}, x_{\max}\}$, $\{y, y_{\min}, y_{\max}\}$ `]`

The x-axis represents the quantity of prey. The y-axis represents the quantity of predators. Solutions $(x, y) = (p(t), q(t))$ to the equations must flow along the arrows given by the differential equations as time progresses.

(4) Qualitatively analyze the relationship between $p(t)$ and $q(t)$. What happens as $p(t)$ increases? As $q(t)$ increases? Are there are any *stable solutions*, meaning that $p(t)$ and $q(t)$ are constant functions of time? What aspects of the relationship change if we increase β? If we increase γ, or δ?

Next, if we think of y as a function of x, we can divide the second differential equation by the first to obtain:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\delta xy - \gamma y}{\alpha x - \beta xy}$$

(5) Apply separation of variables to this equation to find an equation that relates the quantities x and y directly, without any derivatives.

[2] The symbols γ and δ are the Greek letters *gamma* and *delta*.

(6) While it might be difficult to solve for y explicitly as a function of x, we can still graph local solutions using

$$\texttt{ContourPlot[}\ \textit{equation}\ \texttt{,}\ \{x, x_{\text{min}}, x_{\text{max}}\}\texttt{,}\quad \{y, y_{\text{min}}, y_{\text{max}}\}\ \texttt{]}$$

Set $\alpha = \delta = 1$, $\beta = \gamma = 2$, and plot the solutions satisfying the initial conditions:

(i) $(x, y) = (2, 1)$

(ii) $(x, y) = (4, 1)$

(iii) $(x, y) = (4, 4)$

(7) Next, solve the Lotka-Volterra equations numerically using the command:

$$\texttt{NDSolve[}\ \{\textit{diff. equations of } x(t), y(t), \ldots\}\texttt{,}\ \{x, y, \ldots\}\texttt{,}\ \{t, t_{\text{min}}, t_{\text{max}}\}\ \texttt{]}$$

This generates solutions $x = p(t)$, $y = q(t)$ for a finite set of values of t in what Mathematica calls an "interpolating function"[3] Next, use the command

$$\texttt{Plot[\{Evaluate[x[t] /.\%], Evaluate[y[t] /.\%]\},}\ \{t, t_{\text{min}}, t_{\text{max}}\}\ \texttt{]}$$

to plot the graphs of $p(t)$ and $q(t)$.[4] You should do this for each of the initial conditions in the previous problem by specifying the initial condition $(x, y) = (2, 1)$, say, by including "`x[0] == 2, y[0]==1`" in the list of differential equations inside `NDSolve`. What do the graphs show about the relationship between $p(t)$ and $q(t)$?

[3]This means that Mathematica will interpolate based on the known values in order to approximate other values of p and q when asked to do so.

[4]The command " /. " is telling Mathematica to use the solutions that it found previously for x and y. The symbol "`%`" refers to the previous line of output. If you need to refer to an earlier line of output, say from line 37, then you would instead type "`%37`".