MATH 111, EXPLORATION 8

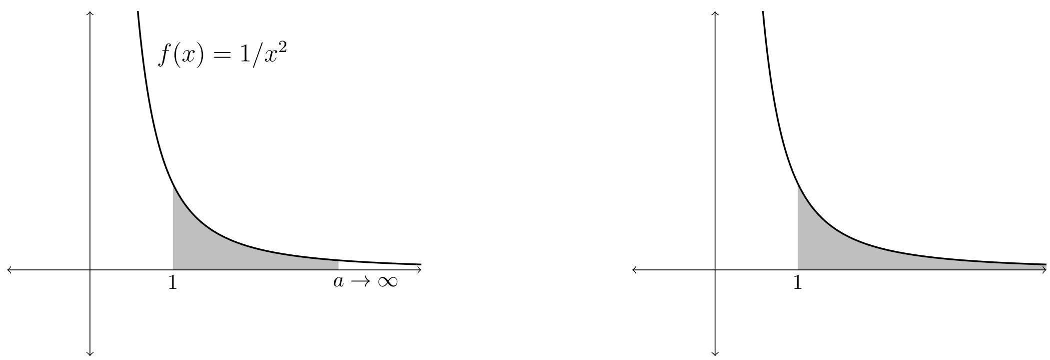
Due Friday, November 17

§7.6 of the textbook is relevant to this exploration.

Consider the area under the graph of $f(x) = 1/x^2$ from $x = 1$ to some arbitrary point $x = a$. By definition, this area is the integral

$$\int_1^a \frac{1}{x^2}\,dx.$$

Now consider sliding the point a off to the right forever, so that it approaches ∞.



The accumulated area under the graph is given by the *improper integral* $\int_1^\infty \frac{1}{x^2}\,dx$. The precise definition is:

$$\int_1^\infty \frac{1}{x^2}\,dx := \lim_{a\to\infty} \int_1^a \frac{1}{x^2}\,dx$$

In Exploration 3, you found that

$$\int_1^a \frac{1}{x^2}\,dx = -\frac{1}{a} + 1.$$

Notice that this also follows from the fundamental theorem of calculus, using the antiderivative $F(x) = -1/x$ for $f(x) = 1/x^2$. This allows us to evaluate the improper integral

$$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{a\to\infty} \int_1^a \frac{1}{x^2}\,dx = \lim_{a\to\infty} \left(-\frac{1}{a} + 1\right) = 1.$$

Therefore, the area of the (unbounded, infinite!) region to the right of $x = 1$ under the graph of $f(x) = 1/x^2$ is 1.

More generally, we define improper integrals of any function $f(x)$ by:

$$\int_a^{\infty} f(x)\,dx := \lim_{b\to\infty} \int_a^b f(x)\,dx$$
$$\int_{-\infty}^{b} f(x)\,dx := \lim_{a\to-\infty} \int_a^b f(x)\,dx$$
$$\int_{-\infty}^{\infty} f(x)\,dx := \lim_{b\to\infty} \int_{-b}^b f(x)\,dx$$

In each case, we say that the improper integral *converges* if the limit converges, and *diverges* if the limit diverges.

In this exploration, you are asked to evaluate the following improper integrals, using whatever methods are useful.

1. $\displaystyle\int_1^{\infty} \frac{1}{x^3}\,dx$
2. $\displaystyle\int_1^{\infty} \frac{1}{x^7}\,dx$
3. $\displaystyle\int_1^{\infty} \frac{1}{x^{3/2}}\,dx$
4. For what values of c does $\displaystyle\int_1^{\infty} \frac{1}{x^c}\,dx$ converge?
5. $\displaystyle\int_{-\infty}^{\infty} e^{-x^2}\,dx$ (Hint: you might more easily recognize the square of this integral)
6. The function $f(x) = 1/\sqrt[4]{x}$ has an asymptote at $x = 0$. What is the area under the graph in between $x = 0$ and $x = 1$?