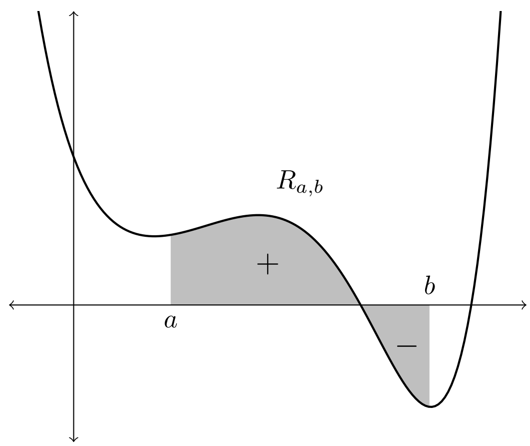
MATH 111, EXPLORATION 7

Due Friday, November 3

Please read §5.1 and §5.2 of the textbook before starting this exploration.

Given a function $f(x)$, let $R_{a,b}$ be the region in between the graph of $f(x)$ and the x-axis for the range of x-values $a \leq x \leq b$. The *signed area* of the region $R_{a,b}$ is the area above the x-axis *minus* the area below the x-axis.

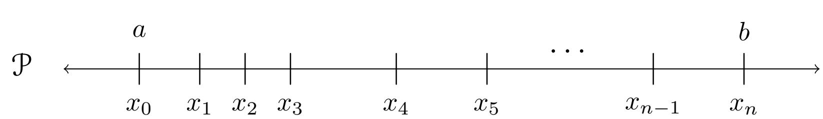


Definition. The *integral* of $f(x)$ from $x = a$ to $x = b$ is the signed area of $R_{a,b}$, as computed using the method of approximation by boxes:

$$\int_a^b f(x)\,dx = \text{area}(R_{a,b}).$$

Sometimes the term "definite integral" is also used for $\displaystyle\int_a^b f(x)\,dx$.

Our goal in this exploration is to unpack this definition and compute some examples. Luckily, we have secretly been computing integrals all semester in the earlier explorations! Let's start by recalling the method of approximation that we have been using. A *partition* $\mathcal{P}$ of the interval $[a, b]$ is a sub-division into n different sub-intervals:



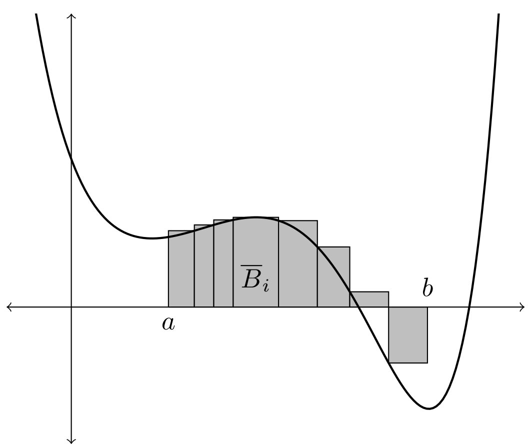
For each integer i in between 1 and n, construct a box $\overline{B}_i$ over the i-th sub-interval $[x_{i-1}, x_i]$ whose top is the largest value of $f(x)$ on $[x_{i-1}, x_i]$. Note that we must take signs into consideration here, so, for example, if the maximum value of $f(x)$ is *negative*, then the "top" of the box is really the bottom, and the box extends from the x-axis downwards. Note that such a box will have *negative* signed area! We define the upper sum associated to the partition $\mathcal{P}$ to be the total signed area of the boxes $\overline{B}_i$:

$$U_{\mathcal{P}} := \sum_{i=1}^{n} \text{area}(\overline{B}_i).$$

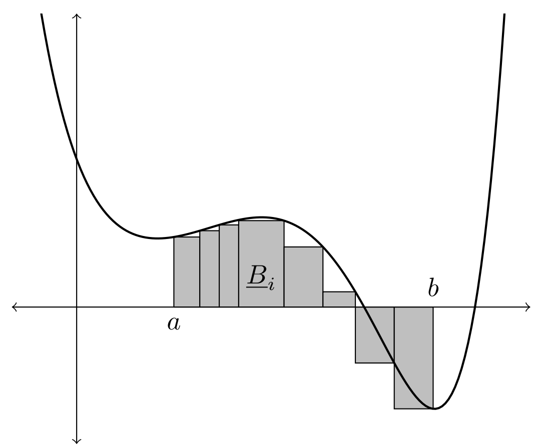
Similarly, for each integer i in between 1 and n, construct a box $\underline{B}_i$ over the i-th sub-interval $[x_{i-1}, x_i]$ whose top is the smallest value of $f(x)$ on $[x_{i-1}, x_i]$. If this number is negative, then we must use the box extending downwards from the x-axis to that y-value, and the resulting box will have negative signed area. We define the lower sum associated to the partition $\mathcal{P}$ to be the total signed area of the boxes $\underline{B}_i$:

$$L_{\mathcal{P}} := \sum_{i=1}^{n} \text{area}(\underline{B}_i).$$

The term *Riemann sum* is used to refer to any sum of area of boxes, such as $U_{\mathcal{P}}$ and $L_{\mathcal{P}}$.



The upper sum $U_{\mathcal{P}}$ is the over-approximation of area$(R_{a,b})$ given by the area of the boxes $\overline{B}_i$ above the graph.



The lower sum $L_{\mathcal{P}}$ is the under-approximation of area$(R_{a,b})$ given by the area of the boxes $\underline{B}_i$ below the graph.

Since the lower sum is an under-approximation and the upper sum is an over-approximation, the inequalities

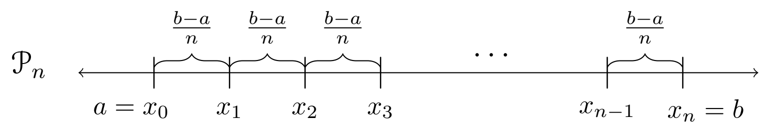
$$L_{\mathcal{P}} \leq \text{area}(R_{a,b}) \leq U_{\mathcal{P}}$$

hold for any partition $\mathcal{P}$. Consider what happens as we sub-divide the partition $\mathcal{P}$ so that it has more and more sub-intervals. The approximations get better and better, and, at least if $f(x)$ is reasonably well-behaved, the numbers $L_{\mathcal{P}}$ and $U_{\mathcal{P}}$ both converge to the true area, which we have defined to be the integral:

$$\lim_{\mathcal{P}} L_{\mathcal{P}} = \int_a^b f(x)\,dx = \lim_{\mathcal{P}} U_{\mathcal{P}}.$$

This formula for the integral is somewhat imprecise, because we have not defined what we mean by "$\lim\limits_{\mathcal{P}}$", the limit over all partitions. There are a few ways to make this precise. We could let Δx be the largest width of a sub-interval in $\mathcal{P}$, and then take the limit as $\Delta x \to 0$. A better approach[1] is to compare the largest value of all of the under-approximations $L_{\mathcal{P}}$ and the smallest value of all of the over-approximations. If they agree, then this common value must be the true area, and so we define the integral to be that number.

Using arbitrary partitions $\mathcal{P}$ gives a good theoretical approach to integration, and this flexibility is useful in certain situations, for example the telescoping sums in Exploration 3. But in practice, it nearly always suffices to use the partition of $[a, b]$ into n equal width sub-intervals:



Notice that the x-coordinates of the partition $\mathcal{P}_n$ are given by

$$x_i = a + i\left(\frac{b-a}{n}\right)$$

Let's use the abbreviations $L_n := L_{\mathcal{P}_n}$ and $U_n := U_{\mathcal{P}_n}$, as we did in previous explorations. The integral, assuming that it exists, is then given by the common value of the limit of the upper and lower sums:

$$\lim_{n\to\infty} L_n = \int_a^b f(x)\,dx = \lim_{n\to\infty} U_n.$$

Notice that if $f(x)$ is *increasing*, meaning that

$$x \le y \quad \Longrightarrow \quad f(x) \le f(y),$$

then the maximum value of $f(x)$ on the i-th subinterval $[x_{i-1}, x_i]$ is obtained at the *right endpoint* x_i. In this case, the upper sum U_n is given by the *right-hand sum*:

$$U_n = \sum_{i=1}^{n} f(x_i)\Delta x \qquad \text{when } f \text{ is increasing on } [a, b].$$

Here $\Delta x = (b-a)/n$ is the width of each subinterval of $\mathcal{P}_n$. Similarly, the minimum value of $f(x)$ on the interval $[x_{i-1}, x_i]$ occurs at the *left endpoint* x_{i-1}. Therefore, the lower sum

[1] Here it is, in full glory:

Definition, Revisited. Suppose that U is the largest number that is smaller than all of the upper sums $U_{\mathcal{P}}$. In other words, $U \le U_{\mathcal{P}}$ for any partition $\mathcal{P}$, and U is the largest number with this property. Similarly, suppose that L is the smallest number that is larger than all of the lower sums $L_{\mathcal{P}}$. In other words, $L \ge L_{\mathcal{P}}$ for any partition $\mathcal{P}$, and L is the smallest number with this property. Since each $L_{\mathcal{P}}$ is an under-approximation and each $U_{\mathcal{P}}$ is an over-approximation, we must have $L \le U$. If $L = U$, then we say that $f(x)$ is *integrable*, and denote this common value by $\int_a^b f(x)\,dx$.

L_n is given by the *left-hand sum*:

$$L_n = \sum_{i=1}^{n} f(x_{i-1})\Delta x \qquad \text{when } f \text{ is increasing on } [a,b].$$

Incidentally, these formulas, which compute the area of the boxes $\overline{B}_i$ and $\underline{B}_i$, are the historical origins of the integral notation. As $n \to \infty$, the width Δx becomes the "infinitesimal" dx and the sum Σ becomes the integral symbol $\int$, which is an old-fashioned way of writing the letter "S".

(1) Draw a picture and explain why U_n is the right-hand sum and L_n is the left-hand sum when $f(x)$ is increasing. Perform a similar analysis when $f(x)$ is *decreasing*, meaning that

$$x \le y \quad \Longrightarrow \quad f(x) \ge f(y).$$

(2) Encode the operation that evaluates the right-hand sum of a function f over the partition $\mathcal{P}_n$ of $[a,b]$ into Mathematica. Note that *functions can be parameters* of other functions in Mathematica, so we can define an operation `RightSum`

`RightSum[f_,a_,b_,n_]:=` (formula for the right-hand sum of f)

which takes as input *any* function in the f position. For example, the right-hand sum for $g(x) = x^2$ on $[1,4]$ with $n = 10,000$ boxes would then be evaluated by typing:

```
g[x_]:=x^2
RightSum[g,1,4,10000]//N
```

(3) Encode the left-hand sum into an operation `LeftSum[f_,a_,b_,n_]` in Mathematica.

(4) Let $f(x) = x$. For each of the following integrals, find the upper and lower sums U_n and L_n with $n = 1,000, 10,000$, and $100,000$ boxes on the indicated interval $[a,b]$. Use your computations, perhaps with even larger values of n if necessary, to determine the value of the integral itself.

(i) $\int_0^1 x\,dx$

(ii) $\int_0^3 x\,dx$

(iii) $\int_{-1}^1 x\,dx$

(iv) $\int_{-1}^3 x\,dx$

(v) $\int_{-5}^3 x\,dx$

(vi) $\int_a^b x\,dx$, for any $a < b$

(5) Do the same for $f(x) = x^2$ (Notice that you've already computed some integrals of x^2 in Exploration 2).

(i) $\int_0^1 x^2\,dx$

(ii) $\int_0^3 x^2\,dx$

(iii) $\int_{-1}^1 x^2\,dx$

(iv) $\int_{-1}^3 x^2\,dx$

(v) $\int_{-5}^3 x^2\,dx$

(vi) $\int_a^b x^2\,dx$, for any $a < b$

(6) Do the same for $f(x) = x^3$.

(i) $\int_0^1 x^3\,dx$

(ii) $\int_0^3 x^3\,dx$

(iii) $\int_{-1}^1 x^3\,dx$

(iv) $\int_{-1}^3 x^3\,dx$

(v) $\int_{-5}^3 x^3\,dx$

(vi) $\int_a^b x^3\,dx$, for any $a < b$

(7) Find $\int_a^b x^4\,dx$, for any $a < b$

(8) Find $\int_a^b x^5\,dx$, for any $a < b$

(9) Do you notice a pattern? What is $\int_a^b x^k\,dx$, for arbitrary $a < b$?

(10) Calculate the following integrals of $f(x) = \cos x$ using the same method. Be careful about where $\cos x$ is increasing or decreasing!

(i) $\int_0^{\pi/6} \cos x\,dx$

(ii) $\int_0^{\pi/4} \cos x\,dx$

(iii) $\int_0^{\pi/3} \cos x\,dx$

(iv) $\int_0^{\pi/2} \cos x\,dx$

(v) $\int_{-\pi/4}^{\pi/3} \cos x\,dx$

(vi) $\int_{-\pi/2}^{2\pi/3} \cos x, dx$

(vii) $\int_{-\pi/2}^{\pi} \cos x\,dx$

(viii) $\int_{-\pi}^{\pi} \cos x\,dx$

What is $\int_a^b \cos x\,dx$, for arbitrary $a < b$?