MATH 111, EXPLORATION 6

Due Friday, October 6

In this exploration, we will consider the exponential function $f(x) = a^x$. We suppose that $a > 0$ is a positive constant, so that the input variable x of the function is in the exponent. Our goal is to determine the derivative $f'(x)$ of the exponential function. It might be useful to consult §1.2, 1.4, 3.2 in the textbook.

The inverse function $f^{-1}(x)$ of the exponential function is the *base a logarithm function* $\log_a(x)$. The logarithm is defined so that

$$\boxed{\log_a(y) = x \qquad \Longleftrightarrow \qquad y = a^x}$$

An equivalent characterization is:

$$\log_a(a^x) = x \qquad \text{and} \qquad a^{\log_a(y)} = y.$$

(1) Use the definition of $\log_a$ and the properties of exponentials to deduce that the logarithm function converts multiplication into addition:

$$\log_a(x \cdot y) = \log_a(x) + \log_a(y) \qquad \text{and} \qquad \log_a(x^y) = y \log_a(x).$$

Recall the re-definition of the derivative function from Sheet 5:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

(2) Let $f(x) = a^x$. Using the formula for the derivative, compute that

$$f'(x) = L(a) \cdot a^x \qquad \text{where } L(a) := \lim_{h \to 0} \frac{a^h - 1}{h}.$$

This means that the derivative of the exponential function is proportional to the exponential function itself, where the constant of proportionality is the number $L(a)$!

(3) Use Mathematica to compute $L(a)$ for a few different values of a (Notice that you already computed $L(2)$ in Exploration 4). Do the numbers $L(a)$ get bigger as a gets bigger? Interpret this in terms of the comment about $L(a)$ as a constant of proportionality above.

Our goal now is to find a number e for which $L(e) = 1$, so that $\frac{d}{dx}(e^x) = e^x$. In order for

$$L(e) = \lim_{h \to 0} \frac{e^h - 1}{h} = 1$$

to hold, the numbers $(e^h - 1)$ and h should be roughly equal, at least when h is very small. In other words,

$$e^h \sim 1 + h, \qquad \text{and so} \qquad e \sim (1+h)^{1/h}.$$

This leads us to make the

Definition. $e := \lim_{h \to 0} (1+h)^{1/h}$.

(4) Use Mathematica to compute the number e to at least ten decimal places. How might you determine the accuracy of your calculation?

(5) Using your answer to (4), verify that $L(e) = 1$. By (2), this means that

$$\boxed{\frac{d}{dx}(e^x) = e^x}$$

so that the exponential function is its own derivative. In fact, the exponential function e^x is the *only* function which is its own derivative!

Let's return to $L(a) = \lim_{h \to 0} (a^h - 1)/h$ for an arbitrary constant a. It is traditional to write ln, and sometimes even just log, for the base e logarithm function $\log_e$. Thus, by definition of the base e logarithm, the equation $a = e^{\ln(a)}$ holds. Substituting this into the formula for $L(a)$, we find that

$$L(a) = \lim_{h \to 0} \frac{(e^{\ln(a)})^h - 1}{h}.$$

(6) Make the substitution $t = h \ln(a)$ and express the above limit in terms of a limit as $t \to 0$. Your answer shouldn't have any instances of h.

(7) Using your answer to (5), find a formula for $L(a)$ in terms of functions that we have already discussed.

(8) What is the derivative of $f(x) = a^x$?

(9) Do problem 40 from §3.2 of the textbook.