MATH 111, EXPLORATION 5

Due Friday, September 29

While approximating area using boxes, we arrived at the idea of the limit ℓ of a sequence of numbers $x_1, x_2, x_3, x_4, \ldots$. We will now discuss the limit ℓ of a function $f(x)$ at a particular x-value a. Intuitively, this means that as x draws arbitrarily close to a, the outputs $f(x)$ become arbitrarily close to ℓ.

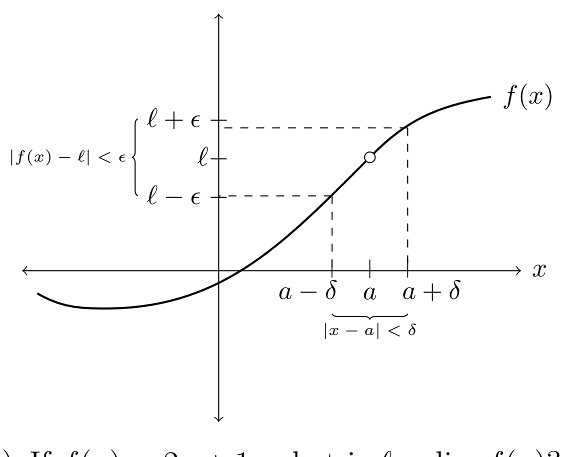
Definition. Suppose that $f(x)$ is a function, and that a and ℓ are numbers. We say that ℓ *is the limit of $f(x)$ as x approaches a* if for any positive amount of closeness ϵ, we can find some positive amount of closeness δ to a wherein x values have outputs $f(x)$ that are within ϵ of ℓ.[1] In other words,

$$\lim_{x\to a} f(x) = \ell \quad \text{MEANS:} \quad \boxed{\begin{array}{c}\text{for any } \epsilon > 0, \text{ there exists } \delta > 0 \text{ such that} \\ \text{if } 0 < |x-a| < \delta, \quad \text{then } |f(x) - \ell| < \epsilon.\end{array}}$$

Remember that

$$\begin{aligned} |f(x) - \ell| < \epsilon \qquad & \text{means that } f(x) \text{ is within } \epsilon \text{ of } \ell, \text{ and} \\ 0 < |x - a| < \delta \qquad & \text{means that } x \text{ is within } \delta \text{ of } a \text{ and } x \neq a. \end{aligned}$$

An important point is that the value $f(a)$ of the function at $x = a$ does not matter in this definition. We never need to consider what happens when $x = a$, only values of x arbitrarily close to a.



Given a target around ℓ (determined by ϵ), we can always make x close enough to a (by finding δ) such that the output $f(x)$ is in the target.

(1) If $f(x) = 2x + 1$, what is $\ell = \lim_{x\to 2} f(x)$? If we want $f(x)$ to be within a distance of $1/10$ from ℓ, how close must x be to 2? Let $\epsilon = 1/10$, and give a value of δ for which

$$0 < |x - 2| < \delta \quad \text{implies that} \quad |f(x) - \ell| < \epsilon. \tag{$\star$}$$

Do the same for $\epsilon = 1/100, 1/1000$, and $1/10000$.

[1] The symbol δ is the Greek letter *delta*. We usually think of it as an extremely small quantity.

(2) If $f(x) = x^2 - x + 3$, what is $\ell = \lim_{x \to 2} f(x)$? Let $\epsilon = 1/10$, and give a value of δ that makes the implication ($\star$) in the definition of the limit hold. Do the same for $\epsilon = 1/100, 1/1000$, and $1/10000$.

In approaching these problems, you might use some of the following commands in Mathematica.

- The command `Abs[` (expression) `]` returns the absolute value of the expression.
- The command `Table[{x,g[x]},{x,a,b,.01}]` produces a list of the values $(x, g(x))$ for $a \le x \le b$ in increments of .01. The value .01 can be replaced by a smaller number to produce x-values that are closer together. The function $g(x)$ can be any function of x. (Hint: what choice of $g(x)$ would be useful in approaching (1) and (2)?)
- The command `ListPlot[` { list of ordered pairs `{x,y}` ... } `]` plots the set of ordered pairs on a pair of coordinate axes. This can be useful in visualizing the data produced using `Table`. Another useful feature is that the term "`%`" always refers to the previous output. So, if you typed `Listplot[%]` immediately after producing a list of data using `Table`, Mathematica will graph that data for you.
- The command `NSolve[` (expression) `==` (expression), `{t}]` solves the given equation for the variable t.
- The command `Plot[{g[x]},{x,-.01,.01},PlotRange->{0,.001}]` graphs the function $g(x)$ over the x-values $-.01 \le x \le .01$ and the y-values $0 \le y \le .001$. These numbers can be changed to suit your needs.

Your computational work gives numerical evidence that the limits are what you think they are. A rigorous proof that a limit is a particular value is a different sort of task. Here, I will *prove* that $\lim_{x \to 2}(2x + 1) = 5$ using the definition of the limit.

Proof. Let $\epsilon > 0$ be an arbitrary positive number. Set $\delta = \epsilon/2$. We will now show that $0 < |x - 2| < \delta$ implies that $|(2x + 1) - 5| < \epsilon$. To do this, assume that $0 < |x - 2| < \delta$. By algebraic manipulation, we then find that

$$\begin{aligned} |(2x+1) - 5| = |2(x-2)| &= 2|x-2| \\ &< 2 \cdot \delta \qquad \text{(by the assumption that } |x-2| < \delta\text{)} \\ &= 2 \cdot \frac{\epsilon}{2} \qquad \text{(by the definition of } \delta\text{)} \\ &= \epsilon. \end{aligned}$$

We have proved the implication in the definition of the limit, and so we conclude that $\lim_{x \to 2}(2x + 1) = 5$. □

These proofs can get rather intricate. Here is a proof that $\lim_{x \to 2}(x^2 - x + 3) = 5$.

Proof. Let $\epsilon > 0$ be given. Choose $\delta = \min\{1, \epsilon/4\}$. To prove the claim, assume that $0 < |x-2| < \delta$. We must show that $|(x^2 - x + 3) - 5| < \epsilon$. By the definition of δ as the minimum of the numbers 1 and $\epsilon/4$, we know that $\delta < 1$. Therefore, by analysis of the behavior of absolute value,

$$\begin{aligned}
|(x-2)+3| &\leq |x-2| + 3 \\
&< \delta + 3 && \text{(by the assumption that } |x-2| < \delta) \\
&< 1 + 3 = 4.
\end{aligned}$$

We will now use this inequality to prove the desired inequality:

$$\begin{aligned}
|(x^2 - x + 3) - 5| &= |x^2 - x - 2| \\
&= |x+1||x-2| \\
&= |(x-2)+3||x-2| \\
&< 4|x-2| && \text{(by the previous inequality)} \\
&< 4\delta && \text{(by the assumption that } |x-2| < \delta) \\
&< 4 \cdot \frac{\epsilon}{4} && \text{(because } \delta < \epsilon/4 \text{ by its definition)} \\
&= \epsilon.
\end{aligned}$$

□

I will not ask you to write proofs like these in Math 111, but you should be aware that such proofs are the correct way to reason rigorously about limits. You will explore this proof method, and much more, if you take Math 112.

(3) Do your values for δ in (1) and (2) agree with the values for δ used in the proofs? Why or why not?

For the remaining problems, find the indicated limit. You should justify your answer using data from Mathematica or a theoretical argument.

(4) $\displaystyle\lim_{x \to 3} \frac{2x^2 - 10x + 12}{x - 3}$

(5) $\displaystyle\lim_{x \to 0} \frac{\sin x}{x}$

(6) $\displaystyle\lim_{x \to 0} \frac{2^x - 1}{x}$