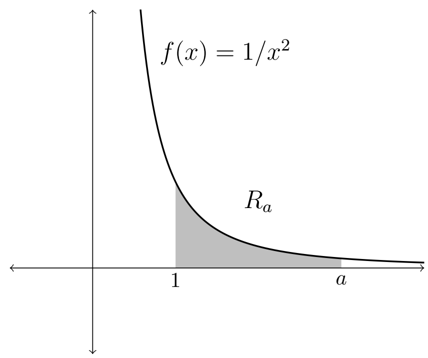
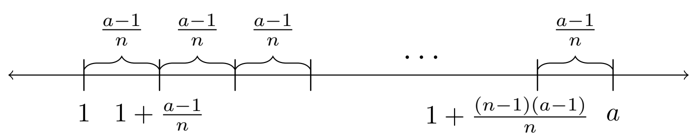
MATH 111, EXPLORATION 3

Due Friday, September 15

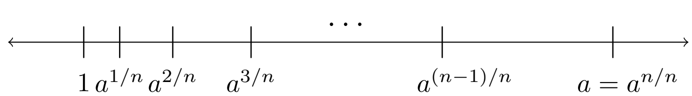
Let $f(x) = 1/x^2$. Our goal is to find the area of the region R_a in the plane bounded by the graph of $f(x)$, the x-axis, the line $x = 1$, and the line $x = a$, where $a > 0$ is a fixed constant.



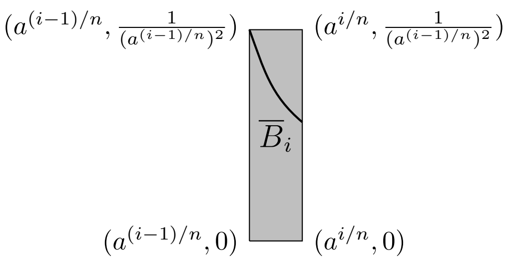
Our strategy will be similar to the one you employed last week to find the area of the region under the curve $y = x^2$. It may be useful to refer to Exploration 2, since we will be using similar notation. Instead of decomposing the interval $1 \le x \le a$ into n subintervals of equal width

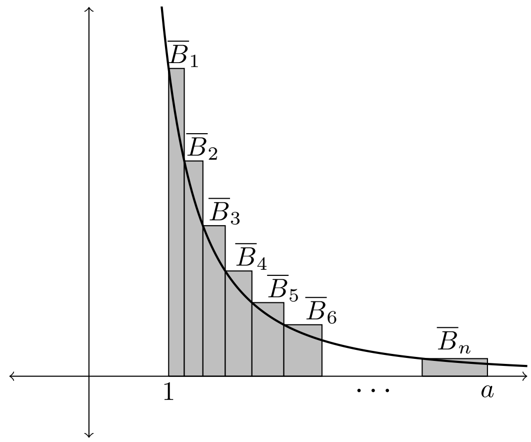


we will use the telescoping decomposition where the endpoints of the subintervals are the points $x = a^{i/n}$ for $i = 0, 1, \ldots, n$:



We will now find an over-approximation of the area of R_a. For each integer i in between 1 and n, construct a box $\overline{B}_i$ over the i-th sub-interval $[a^{(i-1)/n}, a^{i/n}]$ whose top lies over the graph of $f(x) = 1/x^2$ and is the smallest box with this property. Here is the i-th box $\overline{B}_i$, labelled with the coordinates of its corners, and a picture of all of the boxes $\overline{B}_1, \ldots, \overline{B}_n$.





Your first task is to compute the over-approximation of area(R_a) given by the area of the boxes.

(1) What is area($\overline{B}_i$)? Your answer should depend on a, i and n.

(2) What is the n-th upper sum

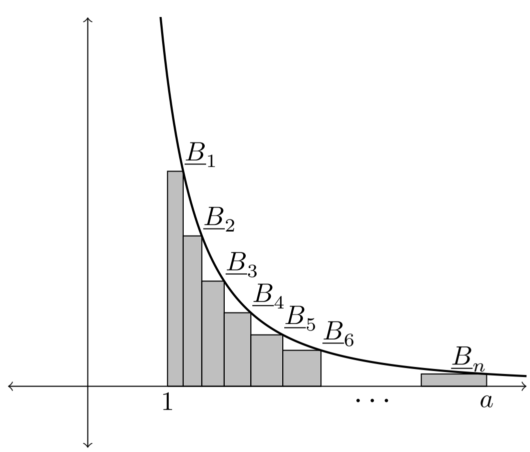
$$U_n := \sum_{i=1}^{n} \text{area}(\overline{B}_i) = \text{area}(\overline{B}_1) + \cdots + \text{area}(\overline{B}_n),$$

in terms of a and n?

In making these calculations, it may be useful to keep in mind the arithmetic rules for manipulating exponents:

$$\boxed{a^{b+c} = a^b \cdot a^c \qquad a^{-b} = \frac{1}{a^b} \qquad (a^b)^c = a^{bc}}$$

We will now find an under-approximation of the area of the region R_a. For each integer i in between 1 and n, construct a box $\underline{B}_i$ over the i-th sub-interval $[a^{(i-1)/n}, a^{i/n}]$ whose top lies under the graph of $f(x) = 1/x^2$ and is the largest box with this property.



(3) What is $\text{area}(\underline{B}_i)$?

(4) What is the n-th lower sum

$$L_n := \sum_{i=1}^{n} \text{area}(\underline{B}_i) = \text{area}(\underline{B}_1) + \cdots + \text{area}(\underline{B}_n),$$

in terms of a and n?

Remember that the upper and lower sums are over and under approximations of the true area of the region R_a, and so they always satisfy the inequalities

$$L_n \leq \text{area}(R_a) \leq U_n$$

regardless of the number n of boxes used. Now, use your answers to (2) and (4) to encode functions $U(a,n)$ and $L(a,n)$ in Mathematica that compute the n-th upper sum U_n and n-th lower sum L_n for the region R_a.

(5) Compute $U(5,10)$, $U(5,100)$, $U(5,1000)$ and $U(5,10000)$.

(6) Compute $L(5,10)$, $L(5,100)$, $L(5,1000)$ and $L(5,10000)$.

(7) What is $\text{area}(R_5)$? What is $\text{area}(R_{10})$? What is $\text{area}(R_{100})$?

(8) What is $\text{area}(R_a)$ in general?

To find $\text{area}(R_a)$ for general a, it might be useful to employ the *geometric series* formula (for a well-chosen value of t)

$$1 + t + t^2 + t^3 + \cdots + \cdots t^n = \frac{1 - t^{n+1}}{1 - t},$$

which follows from the identity

$$(1 - t)(1 + t + t^2 + t^3 + \cdots + t^n) = 1 - t^{n+1}$$

by dividing by $1 - t$.

Now that we know the true area under $f(x) = 1/x^2$, we are also interested in how *good* our approximations L_n and U_n are for various values of n.

(8) How big does n have to be in order for $U(5,n)$ to be within 1/10 of $\text{area}(R_5)$? To be within 1/100, 1/1000, 1/10000 and 1/100000?

(9) How big does n have to be in order for $L(5,n)$ to be within 1/10 of $\text{area}(R_5)$? To be within 1/100, 1/1000, 1/10000 and 1/100000?

As you have observed, the upper sum and lower sum U_n and L_n *tend towards* the true area of the region R_a that they are approximating. We say that area(R_a) *is the limit of* L_n *as* n *approaches* ∞. In symbols, we write this as

$$\lim_{n\to\infty} L_n = \text{area}(R_a).$$

Similarly, for the upper sum we say that *the limit of* U_n *as* n *approaches* ∞ *is* area(R_a), and write:

$$\lim_{n\to\infty} U_n = \text{area}(R_a).$$

The fact that both the over-approximation and the under-approximation converge to the *same limit* is what makes this method of computing area(R_a) work!

An important point is that L_n and U_n are never *equal* to area(R_a) for a finite value of n (at least, not in this example). Rather, they can be made arbitrarily close to area(R_a) by increasing the size of n, as you have shown in exercises (8) and (9). This idea underlies the precise definition of a limit.

Definition. Suppose that $\{x_n\}$ is a sequence of numbers indexed by the positive integers $n = 1, 2, 3, \ldots$, and that ℓ is a number. We say that ℓ *is the limit of* x_n *as* n *approaches* ∞ if for any nonzero amount of closeness ϵ,[1] we can find some point in the sequence after which x_n is always within ϵ of ℓ. In other words:

$$\lim_{n\to\infty} x_n = \ell \quad \overset{\text{(DEF)}}{\iff} \quad \begin{array}{l}\text{for any } \epsilon > 0, \text{ there exists a positive integer } N \\ \text{such that if } n \geq N, \text{ then } |x_n - \ell| < \epsilon.\end{array}$$

(10) Explain how your work in (8) and (9) is an example of the definition of the limit. In each case, what is x_n? What is ℓ? What is ϵ? What is N?

[1] The symbol ϵ is the Greek letter *epsilon*. We usually think of it as an extremely small quantity.